HYPOCONVEXITY AND ESSENTIALLY n-NORMAL OPERATORS

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ABSTRACT. In this paper a classifying structure for the class of essentially n-normal operators on a separable Hilbert space is introduced, and various invariance properties of this classifying structure are studied. The notion of a hypoconvex subset of the algebra $\mathfrak{M}_n$ of all complex $n \times n$ matrices is defined, and it is shown that the set of all equivalence classes of essentially n-normal operators (under a natural equivalence relation), whose reducing essential $n \times n$ matricial spectrum is a given hypoconvex set, forms an abelian group. It is also shown that this correspondence between hypoconvex subsets of $\mathfrak{M}_n$ and abelian groups is a homotopy invariant, covariant functor. This result is then used to prove that Toeplitz operators (on strongly pseudoconvex domains) which have homotopic continuous matricial symbols, are unitarily equivalent up to compact perturbation.

1. Introduction. This paper contains the detailed account of some results already announced in [30] and some further progress that we have made since the publication of that announcement. Our interest in the subject arose from the celebrated paper [10], and in [29] we started a program leading to generalizations of the various important concepts and fruitful ideas contained in [10]. In [26], [27] and [30] we continued our program with a more concrete purpose (see also [13]): to try to obtain a complete set of invariants for the class of essentially n-normal operators on separable Hilbert space up to a reasonable equivalence relation. The present paper might be regarded as a further step toward this goal.

Let $\mathfrak{IC}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathfrak{L}(\mathfrak{IC})$ denote the algebra of all (bounded, linear) operators on $\mathfrak{IC}$. Let $\mathfrak{IC}(\mathfrak{IC})$ denote the (closed) ideal of all compact operators on $\mathfrak{IC}$, and let $\pi_{\mathfrak{IC}}$: $\mathfrak{L}(\mathfrak{IC}) \to \mathfrak{Q}(\mathfrak{IC})$ denote the canonical homomorphism onto the quotient (Calkin) algebra $\mathfrak{Q}(\mathfrak{IC}) = \mathfrak{L}(\mathfrak{IC})/\mathfrak{IC}(\mathfrak{IC})$ (when no confusion arises we will drop the subindex $\mathfrak{IC}$ of π).

DEFINITION 1.1. Two operators T and S in $\mathcal{L}(\mathcal{H})$ are said to be strongly equivalent, and we write $T \sim_s S$, if there exists a unitary operator U in $\mathcal{L}(\mathcal{H})$ such that $U^*TU - S \in \mathcal{K}(\mathcal{H})$. On the other hand, T and S are said to be weakly equivalent, and we write $T \sim S$, if there exists W in $\mathcal{L}(\mathcal{H})$, such that πW is unitary in $\mathcal{Q}(\mathcal{H})$ and $W^*TW - S \in \mathcal{K}(\mathcal{H})$ (see [28]).

These natural equivalence relations on $\mathcal{L}(\mathcal{H})$ were first introduced in [10, Corollary 4.4] where it was proved that for the class of operators with compact self-commutators (i.e. essentially normal operators) these two notions coincide. Furthermore, in [10, Theorem 4.3] it was shown that if two countable families of essentially commuting essentially normal operators are simultaneously weakly equivalent (that is with the same essentially unitary operator), then they are simultaneously strongly equivalent (i.e. with the same unitary operator). An extension of this result will be given in Lemma 3.3.

DEFINITION 1.2. An operator T in $\mathcal{L}(\mathcal{H})$ will be called n-normal if there exists a unitary transformation $U: \mathcal{H} \to \mathcal{H} \otimes \mathbf{C}^n$ such that the operator UTU^* in $\mathcal{L}(\mathcal{H} \otimes \mathbf{C}^n)$ $(= \mathcal{L}(\mathcal{H}) \otimes \mathfrak{M}_n)$ is an $n \times n$ operator matrix with commuting normal entries. Hereafter, $\mathbf{C}^n$ denotes the standard n-dimensional complex vector space and $\mathfrak{M}_n$ denotes the algebra of all complex $n \times n$ matrices (which will be identified with $\mathcal{L}(\mathbf{C}^n)$). An operator T in $\mathcal{L}(\mathcal{H})$ will be called essentially n-normal if there exists a unitary transformation $U: \mathcal{H} \to \mathcal{H} \otimes \mathbf{C}^n$ such that $\pi_{\mathcal{H} \otimes \mathbf{C}^n}(UTU^*)$ is an $n \times n$ matrix with commuting normal entries in $\mathcal{Q}(\mathcal{H})$.

REMARK 1.3. As we shall see in §4, essentially n-normal operators arise naturally in the theory of generalized Toeplitz operators on strongly pseudo-convex domains. In [28] it was shown that two essentially binormal operators in $\mathcal{L}(\mathcal{H})$ may be weakly equivalent but not strongly equivalent. Although the class of essentially n-normal operators is clearly left invariant by strong equivalence, the fact that weak equivalence also preserves essential n-normality is not so obvious and will be proved at the end of the present section.

In [27] it was shown that the reducing essential $n \times n$ spectrum constitutes a complete set of invariants, up to strong equivalence, for the class of n-normal operators in $\mathcal{L}(\mathcal{H})$.

DEFINITION 1.4. Let $T \in \mathcal{L}(\mathcal{H})$ and let $C^*(T)$ denote the C^*-algebra generated by T and $1_{\mathcal{H}}$. The reducing essential $n \times n$ spectrum of T is defined as $R_e^n(T) = \{ L \in \mathfrak{M}_n : \text{there exists a unital } *\text{-homomorphism } \varphi: \pi[C^*(T)] \to \mathfrak{M}_n \text{ such that } \varphi\pi T = L \}$ (see [26] and [13]).

REMARK 1.5. If T is an essentially normal operator in $\mathcal{L}(\mathcal{H})$, then it follows from [29, §6] that $R_e^1(T)$ coincides with the essential spectrum of T. Also, it is easy to show that if T and S are weakly equivalent operators, then $R_e^n(T) = R_e^n(S)$. Notice that, in general, $R_e^n(T) = \varnothing$ for an arbitrary T in $\mathcal{L}(\mathcal{H})$ (see [26, Remark 3.11]), but if T is essentially n-normal, then necessarily $R_e^n(T) \neq \varnothing$ [27, Theorem 3.5]. Thus, the reducing essential $n \times n$ matricial spectrum

constitutes a nontrivial invariant, up to weak or strong equivalence, for the class of essentially n-normal operators in $\mathcal{L}(\mathcal{H})$. Furthermore, it was shown in [27, Theorem 4.4] that if T is an essentially n-normal operator in $\mathcal{L}(\mathcal{H})$, and S is an n-normal operator on $\mathcal{H}$ such that $R_e^n(S) \subset R_e^n(T)$, then $T \oplus S \sim_s T$. In particular, two n-normal operators T and S are strongly equivalent if and only if $R_e^n(T) = R_e^n(S)$.

If T is an arbitrary operator in $\mathcal{L}(\mathcal{H})$ such that $R_e^n(T) = X \neq \varnothing$, then the following properties are easy to verify:

(∗) X is a compact subset of $\mathfrak{M}_n$ which is invariant under unitary conjugation in $\mathfrak{M}_n$.

(∗∗) If $L_j \in X$, $1 \leqslant j \leqslant m$, and P_j is a projection in $\mathfrak{M}_n$ (selfadjoint idempotent) such that $P_j L_j = L_j P_j$, $j = 1, 2, \ldots, m$, and $\sum_{j=1}^m P_j = 1$, then $\sum_{j=1}^m P_j L_j \in X, m = 1, 2, \ldots$.

DEFINITION 1.6. A nonempty subset X of $\mathfrak{M}_n$ which satisfies conditions (∗) and (∗∗) will be called hypoconvex (see [30]). If Y is an arbitrary nonempty compact subset of $\mathfrak{M}_n$, we denote by $\hat{Y}$ the smallest hypoconvex set containing Y and we call it the hypoconvex hull of Y.

LEMMA 1.7. *Let T and S be two essentially n-normal operators in $\mathcal{L}(\mathcal{H})$. Then, $R_e^n(T \oplus S) = [R_e^n(T) \cup R_e^n(S)]^\hat{\ }$.*

PROOF. Since $R_e^n(T \oplus S)$ satisfies conditions (∗) and (∗∗), one inclusion is obvious. For the other inclusion, let $L \in R_e^n(T \oplus S)$ and suppose that L is written on some orthonormal basis of $\mathbf{C}^n$ as the direct sum of its irreducible direct summands $L = \sum_{j=1}^m \oplus L_j$, where $L_j \in \mathfrak{M}_{n_j}$, $\sum_{j=1}^m n_j = n$. Then, for each $j = 1, \ldots, m$, there exists a unital ∗-homomorphism $\varphi_j \colon \pi C^*(T \oplus \mathrm{S}) \to \mathfrak{M}_{n_j}$, such that $\varphi_j \pi(T \oplus S) = L_j$. Since φ_j is an irreducible representation of $C^*[\pi(T \oplus S)]$ on $\mathbf{C}^{n_j}$, from [17, Theorem 2.10.2] φ_j can be dilated to an irreducible representation $\psi_j \colon C^*(T) \oplus C^*(S) \to \mathcal{L}(\mathcal{H}_j')$, for some separable Hilbert space $\mathcal{H}_j'$, $\mathbf{C}^{n_j} \subset \mathcal{H}_j'$, such that $\mathbf{C}_j^n$ reduces $\psi_j A$, and $\psi_j A | \mathbf{C}^{n_j} = \varphi_j A$, for every $A \in C^*[\pi(T \oplus S)]$. Since $1 \oplus 0 \in C^*[\pi(T)] \oplus C^*[\pi(S)]$ and ψ_j is irreducible, we deduce that either $1 \oplus 0$ or $0 \oplus 1$ belongs to $\ker \psi_j$. Suppose, for instance, that $0 \oplus 1 \in \ker \psi_j$, and let θ_j be the restriction of ψ_j to $C^*[\pi(T)]$ (via $1 \oplus 0$). Then $L_j = \varphi_j \pi T = (\theta_j \pi T) | \mathbf{C}^{n_j} \in R_e^{n_j}(T)$, and hence $R_e^{n-n_j}(T) \neq \varnothing$ [12]. It follows that $\sum_{j=1}^m \oplus L_j \in [R_e^n(T) \cup R_e^n(S)]^\hat{\ }$.

DEFINITION 1.8. Let $X \subset \mathfrak{M}_n$ be a hypoconvex set and let $EN_n^s(X)$ [resp., $EN_n(X)$] be the set of all strong [resp., weak] equivalence classes $[T]_s$ [resp., $[T]$] of essentially n-normal operators T in $\mathcal{L}(\mathcal{H})$ such that $R_e^n(T) = X$.

REMARK 1.9. Given two essentially n-normal operators T and S, it readily follows that $T \oplus S$ is also essentially n-normal. Thus, in view of Lemma 1.7, the following abelian operations can be defined on $EN_n^s(X)$ and $EN_n(X)$: $[T]_s + [S]_s = [T \oplus S]_s$ and $[T] + [S] = [T \oplus S]$, where $T \oplus S$ has been identified with an operator in $\mathcal{L}(\mathcal{H})$. From the results stated in Remark 1.5, it

follows that $EN_n^s(X)$ and $EN_n(X)$ are abelian semigroups whose neutral elements consist of the strong equivalence class and weak equivalence class containing all n-normal operators S such that $R_e^n(S) = X$. It is immediate that the correspondence $[T]_s \to [T]$ defines a surjective homomorphism Θ: $EN_n^s(X) \to EN_n(X)$.

The classification problem for essentially n-normal operators, up to weak equivalence [resp., strong equivalence], consists of computing $EN_n(X)$ [resp., $EN_n^s(X)$] for every hypoconvex set X. Although we do not solve the general problem in this paper, we present several interesting partial results, explicitly solving the classification problem for certain hypoconvex subsets of $\mathfrak{M}_n$. In §3 we shall prove that both $EN_n^s(X)$ and $EN_n(X)$ are actually groups, and that ker $\Theta \subset \mathbf{Z}/n\mathbf{Z}$. Furthermore, we shall show that $EN_n^s(\cdot)$ and $EN_n(\cdot)$ are homotopy invariant covariant functors from the category of all hypoconvex subsets of $\mathfrak{M}_n$ into the category of abelian groups. An interesting corollary of this result is that $EN_n^s(\mathfrak{B}_n) = EN_n(\mathfrak{B}_n) = \{0\}$ where $\mathfrak{B}_n$ is the closed unit ball in $\mathfrak{M}_n$. In §4 we use the homotopy invariance of $EN_n^s(X)$ to show that if $\varphi, \psi \colon \overline{\Omega} \to \mathfrak{M}_n$ are two continuous functions on a strongly pseudoconvex domain Ω (with smooth boundary) such that [range $\varphi|\partial\Omega$]⌐ = [range $\psi|\partial\Omega$]⌐ = X, and $\varphi|\partial\Omega$ and $\psi|\partial\Omega$ are in the same homotopy class of $[\partial\Omega, X]$, then the corresponding Toeplitz operators defined by φ and ψ, respectively, are strongly equivalent (that is they are unitarily equivalent up to compact perturbation).

Our next task is to present a direct proof that $EN_n^s(X)$ and $EN_n(X)$ are not empty for every hypoconvex $X \subset \mathfrak{M}_n$. (An alternative proof, using extension theory, will be given in §3.) We shall need the following lemma whose proof (which we omit) can be obtained by using arguments similar to those employed in the proof of [26, Theorem 1.1].

LEMMA 1.10. *Let $\mathfrak{B} \subset \mathcal{L}(\mathcal{H})$ be a unital separable C^*-algebra and suppose there is a family $\{P_k \colon 1 \leq k \leq m\}$ with $m \leq \aleph_0$, of mutually orthogonal projections in $\mathcal{L}(\mathcal{H})$ such that $\sum_{k=1}^m P_k = 1_{\mathcal{H}}$ and $P_k B = B P_k$, for every $B \in \mathfrak{B}$, $1 \leq k \leq m$. If $\varphi \colon \mathfrak{B} \to \mathfrak{M}_n$ is an irreducible representation of $\mathfrak{B}$ on $\mathbf{C}^n$ such that $\mathfrak{B} \cap \mathcal{K}(\mathcal{H}) \subset \ker \varphi$, then there exist n^2 partial isometries $W_{i,j}$ in $\mathcal{L}(\mathcal{H})$, $1 \leq i, j \leq n$, which form a system of matrix units in $\mathcal{L}(\mathcal{H})$ and satisfy:*

(i) *The projection $Q = \sum_{i=1}^n W_{i,i}$ has infinite rank and nullity,*

(ii) *$BQ - QB \in \mathcal{K}(\mathcal{H})$, for every $B \in \mathfrak{B}$,*

(iii) *$QBQ - \sum_{i,j=1}^n [\varphi B]_{i,j} W_{i,j} \in \mathcal{K}(\mathcal{H})$, where $[\varphi B]_{i,j}$ denotes the (i,j)th entry of the matrix φB, for every $B \in \mathfrak{B}$,*

(iv) *$W_{i,j} P_k = P_k W_{i,j}$, for $1 \leq i, j \leq n$, $1 \leq k \leq m$.*

The following theorem generalizes a theorem of Weyl [35].

THEOREM 1.11. *Let $\{L_j\}$ be a bounded sequence in $\mathfrak{M}_n$ and let S be the (diagonal) n-normal operator defined by $S = \sum_{j=1}^{\infty} \oplus L_j$. Then, $R_e^n(S)$ coincides with the hypoconvex hull of the set of limit points of $\{L_j\}$.*

PROOF. If L is a limit point of $\{L_j\}$, then it readily follows from [26, Corollary 3.7] (see also [13, Corollary 4.9]) that $L \in R_e^n(S)$, and since $R_e^n(S)$ is hypoconvex one inclusion follows. In order to prove the opposite inclusion it suffices to show that if M is an irreducible matrix in $R_e^m(S)$, with $1 \leqslant m \leqslant n$, then there exist a sequence of isometries $V_j: \mathbf{C}^m \to \mathbf{C}^n$, $j = 1, 2, \ldots$, and a subsequence $\{L_{k_j}\}$ of $\{L_k\}$ such that $\lim_{j \to \infty} \|M - V_j^* L_{k_j} V_j\| + \|L_{k_j} V_j V_j^* - V_j V_j^* L_{k_j}\| = 0$. Let P_k be the projection from $\mathbf{C}^n \otimes l_2 = \mathbf{C}^n \oplus \mathbf{C}^n \oplus \cdots$ onto the kth copy of $\mathbf{C}^n$, $k = 1, 2, \ldots$. Since $\{P_k\}$ is an orthogonal family of projections on $\mathbf{C}^n \otimes l_2$ such that $\sum_{k=1}^{\infty} P_k = 1$, $P_k S = SP_k$, for every $k = 1, 2, \ldots$, we can apply Lemma 1.10 to the C^*-algebra $C^*(S)$ and to the irreducible m-dimensional representation $\varphi: C^*(S) \to \mathfrak{M}_n$ determined by $\varphi S = M$. Therefore, it follows that there exist m^2 partial isometries $W_{i,j}$ on $\mathbf{C}^n \otimes l_2$, $1 \leqslant i, j \leqslant m$, satisfying the corresponding versions of conditions (i)–(iv) of Lemma 1.10. Let $\{P_{k_j}\}$ be a subsequence of $\{P_k\}$ such that $P_{k_j} W_{1,1} \neq 0$ and let e_j be a unit vector in range $W_{1,1} P_{k_j}$, $j = 1, 2, \ldots$. Also, let $f_1, \ldots, f_m$ be the standard bases for $\mathbf{C}^m$ and let $W_j: \mathbf{C}^m \to \mathbf{C}^n \otimes l_2$ be the linear transformation defined by $W_j f_i = W_{i,1} e_j$, $1 \leqslant i \leqslant m$, $j = 1, 2, \ldots$. Since for each $j = 1, 2, \ldots$ the vectors $W_{1,1} e_j, W_{2,1} e_j, \ldots, W_{m,1} e_j$ are orthonormal, it follows that W_j is an isometry such that range $W_j \subset$ range P_{k_j}. Just as in the proof of [26, Theorem 3.2] we deduce that $\lim_{j \to \infty} (\|SW_j - W_j M\| + \|S^* W_j - W_j M^*\|) = 0$. Now we let $V_j^* = W_j^* | \text{range } P_{k_j}$, $j = 1, 2, \ldots$. Then it follows that $\lim_{j \to \infty} (\|L_{k_j} V_j - V_j M\| + \|L_{k_j}^* V_j - V_j M^*\|) = 0$. This completes the proof of the theorem.

COROLLARY 1.12. *Let $X \subset \mathfrak{M}_n$ be hypoconvex. Then there exists an n-normal operator S such that $R_e^n(S) = X$, and hence $EN_n(X)$ and $EN_n^s(X)$ are not empty.*

PROOF. Let $\{L_j\}$ be a sequence dense in X whose terms are repeated infinitely often, and let $s = \sum_{j=1}^{\infty} \oplus L_j$. Then the corollary is a direct consequence of Theorem 1.11.

We close this section by showing a concrete form that weak equivalence takes on the class of essentially n-normal operators.

THEOREM 1.13. *Let T and S be two operators on $\mathcal{H}$ such that $R_e^n(T) \neq \varnothing$ and $R_e^n(S) \neq \varnothing$. Then $T \sim S$ if and only if there exists an isometry V in $\mathfrak{L}(\mathcal{H})$ with deficiency less than n such that $V^* TV - S \in \mathcal{K}(\mathcal{H})$.*

PROOF. Since isometries with finite deficiencies are essentially unitary (i.e. unitary in the Calkin algebra) one half of the theorem is obvious. In order to prove the other half suppose that there exists W in $\mathcal{L}(\mathcal{H})$ with πW unitary, such that $W^*TW - S \in \mathcal{K}(\mathcal{H})$. From [14] we can assume that W is a partial isometry with finite index k. By perturbing W with a finite rank partial isometry, we can further assume that $\min(\dim \ker W, \dim \ker W^*) = 0$, i.e. that W is an isometry or a coisometry. We suppose now that W is a coisometry (the proof of the case in which W is an isometry is similar and will be omitted). Let j be a positive integer so that $0 < nj - k < n$. Notice that W^*TW is unitarily equivalent to an operator that can be expressed on $[\ker W^*] \oplus [\ker W^*]^\perp$ as $0 \oplus T$. Let $L_1, \ldots, L_j$ be in $R_*^n(T)$. It follows that there exists an isometry V_0 on $[\mathbf{C}^n \otimes \mathbf{C}^j] \oplus \mathcal{H}$, whose range is $\{0\} \oplus \mathcal{H}$, such that $V_0^*[(\Sigma_{i=1}^j \oplus L_i) \oplus T]V_0$ is unitarily equivalent to a compact perturbation of S. Since $(\Sigma_{i=1}^j \oplus L_i) \oplus T \sim_s T$ [27, Theorem 4.4], it readily follows that there exists an isometry V in $\mathcal{L}(\mathcal{H})$ such that $V^*TV - S \in \mathcal{K}(\mathcal{H})$.

THEOREM 1.14. *Let T and S be in $\mathcal{L}(\mathcal{H})$ such that S is an essentially n-normal operator and T is weakly equivalent to S. Then, T is also an essentially n-normal operator.*

PROOF. From Theorem 1.13 there exists an isometry V in $\mathcal{L}(\mathcal{H})$ of deficiency less than n such that $V^*TV - S \in \mathcal{K}(\mathcal{H})$. Thus, T is unitarily equivalent to an operator of the form $0_{\mathfrak{M}} \oplus (S_{ij})$, where $\mathfrak{M}$ is a subspace of $\mathcal{H}$ of dimension less than n and (S_{ij}) is an $n \times n$ operator matrix acting on $\mathfrak{M}^\perp = \mathfrak{N} \otimes \mathbf{C}^n$, with essentially commuting, essentially normal entries in $\mathcal{L}(\mathfrak{N})$. Let $\mathcal{H}' = \mathfrak{M} \oplus \mathfrak{N} \subset \mathcal{H}$ and let $W \in \mathcal{L}(\mathcal{H}')$ be any isometry such that range $W = \mathfrak{M}$. We define the unitary transformation $U: \mathfrak{N} \to \mathcal{H}'$ $(= \mathfrak{M} \oplus \mathfrak{N})$ by $U = W^*|\mathfrak{N}$. We then have $U^*W^*|\mathfrak{N} = 1_{\mathfrak{N}}$, and WU: $\mathfrak{N} \to \mathcal{H}'$ is the natural inclusion. Let $F: \mathfrak{N} \otimes \mathbf{C}^n \to \mathcal{H}' \oplus (\mathfrak{N} \otimes \mathbf{C}^{n-1})$ $(= \mathfrak{M} \oplus (\mathfrak{N} \otimes \mathbf{C}^n) = \mathcal{H})$ be the isometry given by $F = WU \oplus (1_{\mathfrak{N}} \otimes 1_{n-1})$ where 1_{n-1} is the identity matrix in $\mathfrak{M}_{n-1}$. Then, codim range $F =$ codim range $W = \dim \mathfrak{M} < n$. We claim that $F(S_{ij})F^* = 0_{\mathfrak{M}} \oplus (S_{ij})$. Indeed, given $x \in \mathfrak{M}, y \in \mathfrak{N}$ and $z \in \mathfrak{N} \otimes \mathbf{C}^{n-1}$ we have

$$F(S_{ij})F^*(x \oplus y \oplus z)$$
$$= (WU \oplus (1_{\mathfrak{N}} \otimes 1_{n-1}))(S_{ij})(WU \oplus (1_{\mathfrak{N}} \otimes 1_{n-1}))^*(x \oplus y \oplus z)$$
$$= (WU \oplus 1_{\mathfrak{N}} \otimes 1_{n-1})[(S_{ij})(y \oplus z)] = [0_{\mathfrak{M}} \oplus (S_{ij})](x \oplus y \oplus z),$$

and our claim is established. Let $G \in \mathcal{L}(\mathfrak{N} \otimes \mathbf{C}^n)$ $(= \mathcal{L}(\mathfrak{N}) \otimes \mathfrak{M}_n)$ be the essentially unitary operator given by $G = U^*W^*WU \oplus (1_{\mathfrak{N}} \otimes 1_{n-1})$. The preceding remark shows that $G(S_{ij})G^*$ is unitarily equivalent to $0_{\mathfrak{N}} \oplus (S_{ij})$ and

hence unitarily equivalent to the operator T. Let $Z = U^*WU$ and let $R_{ij} = ZS_{ij}Z^*$, $1 \leqslant i, j \leqslant n$. Then the families $\{R_{ij}: 1 \leqslant i, j \leqslant n\}$ and $\{S_{ij}: 1 \leqslant i, j \leqslant n\}$ are simultaneously weakly equivalent via the operator Z (notice that πZ is unitary in $\mathfrak{L}(\mathfrak{N})$), and since each of these families consists of essentially commuting essentially normal operators, from [10, Theorem 4.3], it follows that these two families are simultaneously strongly equivalent, i.e. there exists a unitary Y in $\mathfrak{L}(\mathfrak{N})$ such that $ZS_{ij}Z^* - YS_{ij}Y^* \in \mathcal{K}(\mathfrak{N})$, $1 \leqslant i, j \leqslant n$. In order to complete the proof of the theorem it clearly suffices to prove that the $n \times n$ operator matrix in $\mathfrak{L}(\mathfrak{N}) \otimes \mathfrak{M}_n$ given by

$$(1_{\mathfrak{N}} \oplus (Y \otimes 1_{n-1}))(z \oplus (1_{\mathfrak{N}} \otimes 1_{n-1}))(S_{ij})$$

$$\cdot (Z \oplus (1_{\mathfrak{N}} \otimes 1_{n-1}))^*(1_{\mathfrak{N}} \oplus (Y \otimes 1_{n-1}))^*$$

$$= \begin{bmatrix} ZS_{11}Z^* & ZS_{12}Y^* & \cdots & ZS_{1n}Y^* \\ YS_{21}Z^* & YS_{22}Y^* & \cdots & YS_{2n}Y^* \\ \vdots & \vdots & & \vdots \\ YS_{n1}Z^* & YS_{n2}Y^* & \cdots & YS_{nn}Y^* \end{bmatrix}$$

has essentially commuting essentially normal entries. In order to see that the entries are essentially normal we need only check that, for $1 < j \leqslant n$, the operators $ZS_{1j}Y^*$ and $YS_{j1}Z^*$ are essentially normal, and this is a consequence of the fact that $ZS_{1j}Z^* - YS_{1j}Y^* \in \mathcal{K}(\mathfrak{N})$ and $ZS_{j1}Z^* - YS_{j1}Y^* \in \mathcal{K}(\mathfrak{N})$. In order to show that the entries of the above matrix essentially commute we use a similar argument. For instance,

$$ZS_{11}Z^*YS_{ij}Y^* - YS_{ij}Y^*ZS_{11}Z^* = ZS_{11}Z^*ZS_{ij}Z^* - ZS_{ij}Z^*ZS_{11}Z^*$$

$$+ ZS_{11}Z^*(YS_{ij}Y^* - ZS_{ij}Z^*) - (YS_{ij}Y^* - ZS_{ij}Z^*)ZS_{11}Z^* \in \mathcal{K}(\mathfrak{N}),$$

for $1 < i, j \leqslant n$. The proof of the theorem is complete.

2. The algebra $C_\chi^*(X)$. In this section we investigate some properties of hypoconvex sets which will be needed in the next section, and which seems to be useful for the classification problem of essentially n-normal operators.

We first introduce some terminology. Throughout, unless otherwise specified, all C^*-algebras considered will be unital and separable. Given two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, Hom$(\mathfrak{A}, \mathfrak{B})$ denotes the set of all unital $*$-homomorphisms from $\mathfrak{A}$ into $\mathfrak{B}$. Given a compact Hausdorff space Z, the C^*-algebra $C(Z) \otimes \mathfrak{M}_n$ will be identified with the C^*-algebra of all $n \times n$ continuous matrix valued functions on Z. Also, given a compact subset X of $\mathfrak{M}_n$ we denote by $C_\chi^*(X)$ [30] the C^*-subalgebra of $C(X) \otimes \mathfrak{M}_n$ generated by the identity function $\chi: X \to X$ and the constant function 1 on X. Given a sequence $\{A_j\}$ in the C^*-algebra $\mathfrak{A}$, $C^*(\{A_j\})$ will denote the C^*-subalgebra of $\mathfrak{A}$ generated by $\{A_j\}$ and 1. The following resembles the definition of the Gelfand transform.

DEFINITION 2.1. Let $\mathcal{C}$ be a unital C^*-algebra and let $A \in \mathcal{C}$. Also, let $R^n(A) = \{L \in \mathfrak{M}_n : \text{there exists } \varphi \in \text{Hom}(C^*(A), \mathfrak{M}_n) \text{ such that } \varphi A = L\}$. Let $\Gamma_A : C^*(A) \to C_\chi^*(R^n(A))$ be defined as follows: given $L \in R^n(A)$, let $\varphi_L \in \text{Hom}(C^*(A), \mathfrak{M}_n)$ be the $*$-homomorphism determined by $\varphi_L A = L$. Then, for each $B \in C^*(A)$ we define $\Gamma_A B(L) = \varphi_L B$.

REMARK 2.2. It is easy to check that Γ is a surjective $*$-homomorphism, that is a $*$-isomorphism if $\bigcap_{\varphi \in \text{Hom}(C^*(A), \mathfrak{M}_n)} \ker \varphi = \{0\}$. This occurs, for instance, if $n = 1$ and A is normal, in which case Γ_A coincides with the Gelfand transform, which provides for the continuous functional calculus of $C^*(A)$. Our purpose is to make use of the generalized continuous functional calculus provided by the map Γ_A of Definition 2.1 in order to study certain properties of the C^*-algebra $C_\chi^*(X)$.

LEMMA 2.3. *Let $\mathcal{C} = \mathcal{B} \otimes \mathfrak{M}_n$, where $\mathcal{B}$ is a unital abelian C^*-algebra, and let $A \in \mathcal{C}$. Then $\Gamma_A : C^*(A) \to C_\chi^*(R^n(A))$ is a $*$-isomorphism. Furthermore, if we identify A with a matrix $(B_{i,j})$, with entries $B_{i,j}$ in $\mathcal{B}$, $1 \leq i, j \leq n$, and $Y \subset \mathfrak{M}_n$ is defined as $Y = \{L \in \mathfrak{M}_n : \text{there exists a character } \omega \text{ of } C^*(\{B_{i,j}\}), (\omega B_{i,j}) = L\}$, then $\hat{Y} = R^n(A)$.*

PROOF. To prove the first assertion we need only show that $\bigcap_{\varphi \in \text{Hom}(C^*(A), \mathfrak{M}_n)} \ker \varphi = \{0\}$. Thus, we let $A' \in C^*(A)$, $A' \neq 0$, and we identify A' with the $n \times n$ matrix $(B'_{i,j})$, where $B'_{i,j} \in C^*(\{B_{i,j}\})$, $1 \leq i, j \leq n$. Then, since some $B'_{k,l} \neq 0$ for $1 \leq k, l \leq n$, we can find a character ω of $C^*(\{B_{i,j}\})$ such that $\omega B'_{k,l} \neq 0$. Now, if we let $L = (\omega B_{i,j})$, it follows that $\varphi_L A' \neq 0$. To prove the last statement, let $L \in R^n(A)$ and write L, on some basis of $\mathbf{C}^n$, as

$$(***) \quad L = \sum_{k=1}^m \oplus L_k,$$

where L_k is an irreducible matrix in $\mathfrak{M}_{n_k}$, $\sum_{k=1}^m n_k = n$, $1 \leq k \leq m$.

Let $\psi_k : C^*(A) \to \mathfrak{M}_{n_k}$ be the irreducible representation uniquely determined by $\psi_k A = L_k$, $1 \leq k \leq m$. From [17, Theorem 2.10.2], there exists a dilation $\varphi_k \in \text{Hom}(C^*(\{B_{i,j}\}) \otimes \mathfrak{M}_n, \mathfrak{M}_n)$ of ψ_k such that $\varphi_k A = L_k \oplus L'_k$, for some $L'_k \in \mathfrak{M}_{n-n_k}$, $1 \leq k \leq m$. Furthermore, it follows that there exists a character ω_k of $C^*(\{B_{i,j}\})$ such that $\varphi_k(B'_{i,j}) = (\omega_k B'_{i,j})$, for every $(B'_{i,j}) \in C^*(\{B_{ij}\}) \otimes \mathfrak{M}_n$. In particular, $(\omega_k A) = L_k \oplus L'_k$, $1 \leq k \leq m$, and the proof of the lemma is complete.

COROLLARY 2.4. (a) *Let Z be a compact Hausdorff space and let $f \in C(Z) \otimes \mathfrak{M}_n$. Then $C^*(f)$ is $*$-isomorphic to $C_\chi^*([\text{range } f])$.*

(b) *If Y is a nonempty compact subset of $\mathfrak{M}_n$, then, via the inclusion map $Y \to \hat{Y}$, $C_\chi^*(Y)$ is $*$-isomorphic to $C_\chi^*(\hat{Y})$.*

PROOF. It is immediate from Lemma 2.3.

In the following lemma we list some functorial properties of hypoconvexity.

LEMMA 2.5. *Let X, Y be two nonempty compact subsets of $\mathfrak{M}_n$.*

(a) *Let $f \in C_\chi^*(X)$ such that* range $f \subset Y$ *and define f^*: $C_\chi^*(Y) \to C_\chi^*(X)$ by $f^*h = h \circ f$, for $h \in C_\chi^*(Y)$. Then f^* is a unital $*$-homomorphism. Furthermore, if $g \in C_\chi^*(Y)$, then $g \circ f \in C_\chi^*(X)$ and $(g \circ f)^* = f^*g^*$.*

(b) *If Y is hypoconvex, Z is a compact Hausdorff space, $\rho \in \text{Hom}(C_\chi^*(Y), C(Z) \otimes \mathfrak{M}_n)$ and $f = \rho\chi$, then* range $f \subset Y$ *and $\rho = f^*$, where f^* is defined as in (a).*

(c) *If X and Y are hypoconvex, ρ: $C_\chi^*(Y) \to C_\chi^*(X)$ is a $*$-isomorphism and $f = \rho\chi$, then f is bijective, $C^*(f) = C_\chi^*(X)$, f^{-1}: $Y \to X$ is in $C_\chi^*(Y)$, and $(f^{-1})^* = \rho^{-1}$.*

PROOF. Part (a) is easy and the only thing that needs some checking in part (b) is the fact that range $f \subset Y$. To this end, let $z \in Z$ and let $\varphi \in \text{Hom}(C_\chi^*(Y), \mathfrak{M}_n)$ given by $\varphi h = \rho h(z)$, $h \in C_\chi^*(Y)$. On some basis of $\mathbf{C}^n$, we decompose φ in its irreducible subrepresentations $\varphi = \Sigma_{k=1}^m \oplus \varphi_k$, where $\varphi_k : C_\chi^*(Y) \to \mathfrak{M}_{n_k}$, $1 \leqslant k \leqslant m$, $\Sigma_{k=1}^m n_k = n$. Employing [17, Theorem 2.10.2], we deduce that for each $k = 1, 2, \ldots, m$, there exists an irreducible representation ψ_k: $C(Y) \otimes \mathfrak{M}_n \to \mathfrak{M}_n$, that satisfies $\psi_k h = \varphi_k h \oplus \varphi_k' h$ for every $h \in C_\chi^*(Y)$ and some $\varphi_k' \in \text{Hom}(C_\chi^*(Y), \mathfrak{M}_{n-n_k})$. It follows that there exists $M_k \in Y$ such that, for every $h \in C(Y) \otimes \mathfrak{M}_n$, $\psi_k h = h(M_k)$, $1 \leqslant k \leqslant m$. In particular, $M_k = \chi M_k = \varphi_k \chi \oplus \varphi_k' \chi$. Since Y is hypoconvex, it follows that $\varphi\chi = \Sigma_{k=1}^m \varphi_k \chi \in Y$, and hence $f(z) = \rho\chi(z) = \varphi\chi \in Y$. Now, the validity of statement (b) clearly follows. In order to prove part (c) let $f = \rho\chi$ and $g = \rho^{-1}\chi$ which lie in $C_\chi^*(Y)$ and $C_\chi^*(X)$, respectively. In order to complete the proof of the lemma it suffices to show that $g \circ f = \chi_X$ and $f \circ g = \chi_Y$ (we check only the first one since the other one follows by symmetry). Let $p_j(\chi_Y, \chi_Y^*)$, $j = 1, 2, \ldots,$ be a sequence of polynomials on the noncommutative variables χ_Y and χ_Y^* that converges to g. Then

$$\chi_X = \rho\rho^{-1}\chi_X = \rho g = \lim_{j \to \infty} (\chi_Y, \chi_Y^*) = \lim_{j \to \infty} p_j(f, f^*) = g \circ f.$$

Notice that this also proves that $\chi_X \in C^*(f)$. The proof of the lemma is complete.

DEFINITION 2.6. For each $n = 1, 2, \ldots$ we denote by $\mathbf{HC}_n$ the category of all hypoconvex subsets of $\mathfrak{M}_n$, where $\text{Hom}(X, Y) = \{f \in C_\chi^*(X)$: range $f \subset Y\}$, $X, Y \in \mathbf{HC}_n$. Functions in $\text{Hom}(X, Y)$ will be called hypoconvex.

REMARK 2.7. (a) Lemma 2.5 tells us that the correspondence $X \to C_\chi^*(X)$ defines a completely faithful contravariant functor from $\mathbf{HC}_n$ into the category of unital separable C^*-algebras all of whose irreducible representations are finite dimensional.

(b) Let $X, Y \in \mathbf{HC}_n$ and let $f \in \text{Hom}(X, Y)$. Also, let $L_k \in X$ and let P_k be a projection in $\mathfrak{M}_n$ such that $P_k L_k = L_k P_k$, $1 \leqslant k \leqslant m$, and such that $\Sigma_{k=1}^m P_k = 1$. Then $P_k f(L_k) = f(L_k) P_k$ and $f(\Sigma_{k=1}^m P_k L_k) = \Sigma_{k=1}^m P_k f(L_k)$.

Therefore, if $X' \subset X$, then $[f(X')\hat{\ }]\hat{\ } = [f(X')]\hat{\ }$. Furthermore, if p is a polynomial in χ (or χ^*), then $[p(X')]\hat{\ } = p[(X')\hat{\ }]$. On the other hand, if f^*: $C_\chi^*(Y) \to C_\chi^*(X)$ is an isomorphism and $(X')\hat{\ } = X$, then $[f(X')]\hat{\ } = Y$; indeed, $[f(X')]\hat{\ } = [f(X')\hat{\ }]\hat{\ } = [f(X)]\hat{\ } = \hat{Y} = Y$.

DEFINITION 2.8. From now on we shall denote by $\mathcal{I}\mathfrak{M}_n$ the set of irreducible matrices in $\mathfrak{M}_n$. Given a hypoconvex $X \subset \mathcal{I}\mathfrak{M}_n$, we say that a compact subset X' of X is generating for X, if X' consists of inequivalent matrices and $(X')\hat{\ } = X$.

THEOREM 2.9. *Let X be a hypoconvex set in $\mathcal{I}\mathfrak{M}_n$. Then the following statements are equivalent.*

(a) *There exists a generating subset X' for X.*

(b) *There exists a nonempty compact subset X' of X such that $C_\chi^*(X)$ is *-isomorphic to $C(X') \otimes \mathfrak{M}_n$.*

(c) *$C_\chi^*(X)$ is *-isomorphic to $C(Z) \otimes \mathfrak{M}_n$ for some compact Hausdorff space Z.*

PROOF. To see that (a) implies (b), first notice that from Corollary 2.4(b) it follows that $C_\chi^*(X)$ is *-isomorphic to $C_\chi^*(X')$. Now, from the generalized Stone-Weierstrass theorem [17, Lemma 11.1.4] and the fact that X' consists of inequivalent irreducible matrices we see that $C_\chi^*(X') = C(X') \otimes \mathfrak{M}_n$. Thus, we have proved that (a) implies (b). Since (b) implies (c) is obvious, we turn to the proof that (c) implies (a). Suppose that $\rho: C_\chi^*(X) \to C(Z) \otimes \mathfrak{M}_n$ is a *-isomorphism and let $f = \rho\chi$. From Lemma 2.5(b) we deduce that range $f \subset X$ and since $C^*(f) = C^*(\rho\chi) = \rho C_\chi^*(X) = C(Z) \otimes \mathfrak{M}_n$, it follows that range f consists of inequivalent matrices. Let $Y = [\text{range } f]\hat{\ }$, and let $\sigma: C_\chi^*(X) \to C_\chi^*(Y)$ be the *-isomorphism given by $\sigma = \Gamma_f \circ \rho$ where Γ_f is defined as in Definition 2.1. We deduce that if we set $g = \sigma\chi$, then $g: Y \to X$ is bijective and $g^* = \sigma$. But, $g = \sigma\chi_X = \Gamma_f\rho\chi_X = \Gamma_f f = \chi_Y$. Since g is bijective $\chi_Y = \chi_X$, and hence $X = Y$. In particular, $[\text{range } f]\hat{\ } = X$ and therefore range f is a generating set for X. The proof of the theorem is complete.

REMARK 2.10. (a) If X is a hypoconvex subset of $\mathcal{I}\mathfrak{M}_2$, then there exists a canonical generating set $X^\sim$ for X. In fact, it follows from [24, Theorem 3.1] that if $\approx$ denotes the relation of unitary equivalence on $\mathfrak{M}_n$, and p_n: $\mathfrak{M}_n \to \mathfrak{M}_n/\approx$ denotes the quotient map, then there exists a continuous cross-section $s: \mathfrak{M}_2/\approx \to \mathfrak{M}_2$ of p_2, i.e., $p_2 \circ s$ is the identity on $\mathfrak{M}_2/\approx$. Thus, we simply set $X^\sim = s \circ p_2(X)$ for every hypoconvex $X \subset \mathcal{I}\mathfrak{M}_2$.

(b) For $n > 2$ in general there are hypoconvex subsets of $\mathcal{I}\mathfrak{M}_n$ with no generating subsets for them. Indeed, this is an easy consequence of the proof of [24, Theorem 3.2] where it is shown that there exists no continuous cross-section of the quotient map $p_n: \mathfrak{M}_n \to \mathfrak{M}_n/\approx$, for $n > 2$.

(c) If X is a hypoconvex subset of $\mathfrak{M}_n$ which does not consist only of

irreducible matrices, then it is easy to see that $C_X^*(X)$ cannot be $*$-isomorphic to $C(Z) \otimes \mathfrak{M}_n$, for any compact Hausdorff space Z.

We are indebted to Vern Paulsen who pointed out a mistake in an earlier version of the above remark.

3. Extensions of C^*-algebras. In this section we explain how the classification problem for essentially n-normal operators can be rephrased in terms of certain classification problems in the theory of C^*-algebra extensions of the ideal of compact operators on $\mathcal{K}$. A similar formulation was already used in [10] where extension theory was used to solve the classification problem for essentially normal operators. We now recall some definitions and certain background material that will be needed in the sequel.

For a (unital, separable) C^*-algebra $\mathcal{C}$ we let $E(\mathcal{C})$ be the set of all unital $*$-monomorphisms $\tau: \mathcal{C} \to \mathcal{Q}(\mathcal{K})$. We say that τ and τ' in $E(\mathcal{C})$ are weakly [resp., strongly] equivalent, and we write $\tau \sim \tau'$ [resp., $\tau \sim_s \tau'$] if there exists W in $\mathcal{L}(\mathcal{K})$, πW unitary [resp., W unitary] such that $\pi W^*(\tau A)\pi W = \tau' A$ for every $A \in \mathcal{C}$. For each τ in $E(\mathcal{C})$ we denote by $[\tau]$ and $[\tau]_s$ the weak and strong equivalence classes containing τ, respectively, and we denote by $\mathrm{Ext}(\mathcal{C})$ and $\mathrm{Ext}^s(\mathcal{C})$ the set of all weak equivalence classes and strong equivalence classes of $E(\mathcal{C})$, respectively. We define the following natural abelian operations on $\mathrm{Ext}(\mathcal{C})$ and $\mathrm{Ext}^s(\mathcal{C})$: $[\tau] + [\tau'] = [\tau \oplus \tau']$ and $[\tau]_s + [\tau']_s = [\tau \oplus \tau']_s$, τ and τ' in $E(\mathcal{C})$, where $\tau \oplus \tau': \mathcal{C} \to \mathcal{Q}(\mathcal{K}) \oplus \mathcal{Q}(\mathcal{K})$ has been identified with an element of $E(\mathcal{C})$ via an identification of $\mathcal{Q}(\mathcal{K}) \oplus \mathcal{Q}(\mathcal{K})$ with a C^*-subalgebra of $\mathcal{Q}(\mathcal{K})$. We shall say that $\tau \in E(\mathcal{C})$ is trivial if there exists $\sigma \in \mathrm{Hom}(\mathcal{C}, \mathcal{L}(\mathcal{K}))$ such that $\tau = \pi\sigma$ and the set of all trivial elements in $E(\mathcal{C})$ will be denoted by $E_0(\mathcal{C})$.

One of the most remarkable generalizations of the work of Brown, Douglas and Fillmore [10] was the discovery ([34] and [16]) that if $\mathcal{C}$ is a nuclear C^*-algebra then $\mathrm{Ext}(\mathcal{C})$ and $\mathrm{Ext}^s(\mathcal{C})$ form abelian groups whose identities consist of the weak equivalence class and strong equivalence class containing $E_0(\mathcal{C})$, respectively (see also [3]). In [10] it was shown that for every compact metric space X, $\mathrm{Ext}[C(X)] = \mathrm{Ext}^s[C(X)]$. For this reason we shall denote by $\mathrm{Ext}(X)$ either of the sets $\mathrm{Ext}[C(X)]$ or $\mathrm{Ext}^s[C(X)]$ for a compact metric space X. They also showed that $\mathrm{Ext}(\cdot)$ is a homotopy invariant covariant functor from the category of compact metric spaces into the category of abelian groups, and in [31] this result was generalized to the category of quasi-diagonal nuclear separable C^*-algebras. Given two C^*-algebras $\mathcal{C}$ and $\mathcal{B}$ we shall endow $\mathrm{Hom}(\mathcal{C}, \mathcal{B})$ with the strong operator topology. Also, if $\zeta \in \mathrm{Hom}(\mathcal{C}, \mathcal{B})$ we define $\zeta_*: \mathrm{Ext}(\mathcal{B}) \to \mathrm{Ext}(\mathcal{C})$ and $\zeta_*^s: \mathrm{Ext}^s(\mathcal{B}) \to \mathrm{Ext}^s(\mathcal{C})$ by $\zeta_*[\tau] = [(\tau \circ \zeta) \oplus \tau_0]$ and $\zeta_*^s[\tau]_s = [(\tau \circ \zeta) \oplus \tau_0]_s$, respectively, where $\tau_0 \in E_0(\mathcal{C})$. One of the corollaries of the main result in [31] is the following homotopy theorem.

THEOREM 3.1. *Let $\mathcal{C}$ and $\mathcal{B}$ be unital separable C^*-algebras all of whose irreducible representations are finite dimensional, and let θ: $[0, 1] \to$ Hom($\mathcal{C}, \mathcal{B}$) be a continuous function, then $[\theta(0)]_* = [\theta(1)]_*$ and $[\theta(0)]_*^r = [\theta(1)]_*^r$.*

REMARK 3.2. The map Φ: $\text{Ext}^r(\mathcal{C}) \to \text{Ext}(\mathcal{C})$ given by $\Phi[\tau]_s = [\tau]$ for $\tau \in E(\mathcal{C})$ is a surjective homomorphism but in general is not injective. Also, Φ is a natural transformation from the contravariant functor $\text{Ext}^r(\cdot)$ to the contravariant functor $\text{Ext}(\cdot)$ on the category of unital separable C^*-algebras. As pointed out in [9] there is a natural action of the integers $\mathbf{Z}$ on $\text{Ext}^r(\mathcal{C})$ given by $\alpha n([\tau]_s) = [\pi W^* \tau \pi W]_s$, where W has index n and πW is unitary in $\mathfrak{A}(\mathcal{K})$, $n \in \mathbf{Z}$. It follows that $\ker \Phi$, which is the weak equivalence class containing $E_0(\mathcal{C})$, is the orbit of the action α on the neutral element of $\text{Ext}^r(\mathcal{C})$.

The above action α defines a surjection η: $\mathbf{Z} \to \ker \Phi$. More precisely, η can be defined as follows: Let $\tau_0 \in E_0(\mathcal{C})$ and let U^+ be a unilateral shift in $\mathcal{L}(\mathcal{K})$; then for $k \in \mathbf{Z}$, we let $\eta(k) = [(\pi U^{+k})\tau_0(\pi U^{+*k})]$. Since U^{+k} is unitarily equivalent to $U^+ \oplus U^+ \oplus \cdots \oplus U^+$ (k-times), it readily follows that η is a homomorphism.

The next result (which was announced in [30]) shows that kernel of Φ can be explicitly computed in certain cases. We are indebted to the referee of this paper for having suggested a simplification in our original proof of the following lemma.

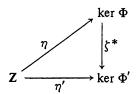LEMMA 3.3. *Let $\mathcal{C}$ be a (unital, separable) C^*-algebra.*

(a) *If there exists an n-dimensional irreducible representation of $\mathcal{C}$, then $\ker \Phi$ is a homomorphic image of $\mathbf{Z}_n$ ($= \mathbf{Z}/n\mathbf{Z}$).*

(b) *If there exists a unital inclusion j: $\mathfrak{M}_n \to \mathcal{C}$, then $\mathbf{Z}_n$ is a homomorphic image of $\ker \Phi$. In particular, if $\mathcal{C} = \mathcal{B} \otimes \mathfrak{M}_n$, for some abelian C^*-algebra $\mathcal{B}$, then $\ker \Phi \cong \mathbf{Z}_n$.*

PROOF. Let η: $\mathbf{Z} \to \ker \Phi$ be as in the above remark. In order to prove part (a), it clearly suffices to show that $\eta(1)$ has order at most n. Thus, let σ_0: $\mathcal{C} \to \mathcal{L}(\mathcal{K})$ be a faithful unital representation so that $\pi\sigma_0(= \tau_0) \in E_0(\mathcal{C})$. Also, let $P = 1 - U^{+n}U^{+*n}$, and let φ: $\mathcal{C} \to \mathcal{L}(P\mathcal{K})$ be an n-dimensional irreducible representation of $\mathcal{C}$ (notice that $\dim(P\mathcal{K}) = n$). It easily follows that the map σ: $\mathcal{C} \to \mathcal{L}(\mathcal{K})$ defined by $\sigma(A) = U^{+n}(\sigma_0 A)U^{+*n} + (\varphi A) \circ P$ is a unital faithful representation of $\mathcal{C}$. Moreover,

$$n\eta(1) = \eta(n) = \left[(\pi U^{+n})\pi\sigma_0(\pi U^{+*n}) \right]$$
$$= \left[\pi(U^{+n}\sigma_0 U^{+*n} + \varphi P) \right] = \left[\pi\sigma \right] = 0,$$

as desired. In order to prove (b) we first observe that if $\mathcal{C}'$ is another C^*-algebra and ζ in Hom($\mathcal{C}', \mathcal{C}$) is injective, then for every τ_0 in $E_0(\mathcal{C})$, it

follows that $\tau_0\zeta$ is in $E_0(\mathcal{C}')$, and hence the following diagram is commutative.

$$\begin{array}{ccc}
& & \ker\Phi \\
& \nearrow^{\eta} & \downarrow{\zeta^*} \\
Z & \xrightarrow[\eta']{} & \ker\Phi'
\end{array}$$

where the definitions of the maps Φ' and η' are similar to those of Φ and η, respectively. The proof of the lemma is completed by applying the preceding remark to the map $\zeta = j$, and the C^*-algebra $\mathcal{C}' = \mathfrak{M}_n$, and observing $\ker\Phi' = \mathrm{Ext}^s(\mathfrak{M}_n) \simeq \mathbf{Z}_n$ (cf. [**19**]).

LEMMA 3.4. *Let $\mathcal{C}$ be as above and let n be a positive integer. Also, let Ψ:* $\mathrm{Ext}(\mathcal{C}) \to \mathrm{Ext}(\mathcal{C} \otimes \mathfrak{M}_n)$ *be given by $\Psi[\tau] = [\tau \otimes 1_n]$. Then Ψ is an isomorphism.*

PROOF. Since it is obvious that Ψ is a homomorphism we prove that Ψ is bijective by exhibiting an inverse Ψ': $\mathrm{Ext}(\mathcal{C} \otimes \mathfrak{M}_n) \to \mathrm{Ext}(\mathcal{C})$ of Ψ. Let $\tau \in E(\mathcal{C} \otimes \mathfrak{M}_n)$ and let $\{e_{i,j}: 1 \leqslant i, j \leqslant n\}$ be the natural system of matrix units in $\mathcal{C} \otimes \mathfrak{M}_n$. Then, $\{\tau e_{i,j}: 1 \leqslant i, j \leqslant n\}$ is a system of matrix units in $\mathfrak{L}(\mathfrak{K})$ such that $\sum_{i=1}^{n} \tau e_{i,i} = 1$. Reasoning, for instance, as in [**26**, Lemma 3.1], there exists a system of matrix units $\{W_{i,j}: 1 \leqslant i, j \leqslant n\}$ in $\mathfrak{L}(\mathfrak{K})$ such that $\pi W_{i,j} = \tau e_{i,j}$ and the subspace $\mathfrak{K}' = \mathrm{range}\ \sum_{i=1}^{n} W_{i,i}$ has codimension less than n in $\mathfrak{K}$. Now, let V be any isometry in $\mathfrak{L}(\mathfrak{K})$ such that range $V = \mathfrak{K}'$. Let $V_{i,j} = V^* W_{i,j} V$, and let $\tau' \in E(\mathcal{C} \otimes \mathfrak{M}_n)$ be defined by $\tau' B = \pi V^* \tau B\pi V$, $B \in \mathcal{C} \otimes \mathfrak{M}_n$. We deduce that $\{V_{i,j}\}$ is a system of matrix units in $\mathfrak{L}(\mathfrak{K})$ such that $\sum_{i=1}^{n} V_{i,i} = 1_{\mathfrak{K}}$ and $\tau' e_{i,j} = \pi V_{i,j}$, $1 \leqslant i, j \leqslant n$. It readily follows that there exists a unitary transformation U: $\mathfrak{K} \to \mathfrak{K} \otimes \mathbf{C}^n$ such that $UV_{i,j}U^*$ is the $n \times n$ operator matrix in $\mathfrak{L}(\mathfrak{K} \otimes \mathfrak{M}_n)$, all of whose entries are zero except the (i, j)th entry which is $1_{\mathfrak{K}}$. Via the unitary U, the $*$-monomorphism τ': $\mathcal{C} \otimes \mathfrak{M}_n \to \mathfrak{L}(\mathfrak{K})$ can be identified with a $*$-monomorphism, which we still call τ', from $\mathcal{C} \otimes \mathfrak{M}_n$ into $\mathfrak{L}(\mathfrak{K}) \otimes \mathfrak{M}_n$. We define now the positive linear map $\bar{\tau}$: $\mathcal{C} \to \mathfrak{L}(\mathfrak{K})$ as follows: For each $A \in \mathcal{C}$, we let $\bar{\tau}A$ be the 1, 1 entry of the matrix in $\mathfrak{L}(\mathfrak{K}) \otimes \mathfrak{M}_n$ given by $\tau'(A \otimes e_{1,1})$. In view of the above identifications, it follows that $\bar{\tau}$ is a unital $*$-monomorphism, and that

$$\tau'(A \otimes e_{i,j}) = \tau'((1 \otimes e_{i,1})(A \otimes e_{1,1})(1 \otimes e_{1,j}))$$

$$= (1 \otimes e_{i,1})((\bar{\tau}A) \otimes e_{1,1})(1 \otimes e_{1,j}) = (\bar{\tau}A) \otimes e_{i,j},$$

$1 \leqslant i, j \leqslant n$, $A \in \mathcal{C}$. Therefore, given (A_{ij}) in $\mathcal{C} \otimes \mathfrak{M}_n$, we have $\tau'(A_{ij}) = \sum_{i,j} \tau'(A_{ij} \otimes e_{i,j}) = \sum_{i,j}(\bar{\tau}A_{ij}) \otimes e_{i,j} = (\bar{\tau} \otimes 1_n)(A_{ij})$, that is, $\tau' = \bar{\tau} \otimes 1_n$. Suppose now that $\bar{\tau} \otimes 1_n \sim \tau_1 \otimes 1_n$ for some τ_1 in $E(\mathcal{C})$. This means that

there exists an essentially unitary W in $\mathcal{L}(\mathcal{K}) \otimes \mathfrak{M}_n$ such that $(\pi W)(\tilde{\tau} \otimes 1_n)$ $= (\tau_1 \otimes 1_n)(\pi W)$. In particular, $(\pi W)(1 \otimes e_{i,j}) = (1 \otimes e_{i,j})(\pi W)$, $1 \leq i, j \leq n$. It readily follows that there exists an essentially unitary W' in $\mathcal{L}(\mathcal{K})$ such that $W = W' \otimes 1_n$, and $(\pi W')\tilde{\tau} = \tau_1(\pi W')$, and hence $\tilde{\tau} \sim \tau_1$. Now we define the map Ψ' by letting $\Psi'[\tau] \, (= \Psi'[\tau']) = [\tilde{\tau}]$. The above remarks show that Ψ' is well defined and it is the inverse of Ψ, as required.

LEMMA 3.5. *Let X be a nonempty compact subset of $\mathfrak{M}_n$. If $\tau \in E(C(X) \otimes \mathfrak{M}_n)$ and T is an operator in $\mathcal{L}(\mathcal{K})$ such that $\pi T = \tau \chi$, then T is essentially n-normal and $R_e^n(T) = \hat{X}$. Conversely, if X is hypoconvex and T is any essentially n-normal operator such that $R_e^n(T) = X$, then there exists a τ in $E(C(X) \otimes \mathfrak{M}_n)$ such that $\tau \chi = \pi T$.*

PROOF. In order to prove the first part of the theorem, we let $\tau': C(X) \otimes \mathfrak{M}_n \to \mathfrak{Q}(\mathcal{K}) \otimes \mathfrak{M}_n$, and $\tilde{\tau}: C(X) \to \mathfrak{Q}(\mathcal{K})$ be the unital $*$-monomorphisms induced by τ, as in the proof of Lemma 3.4, where $\mathcal{Q} = C(X)$, so that $\tau' = \tilde{\tau} \otimes 1_n$ and $\tau' \sim \tau$. Also let T' in $\mathcal{L}(\mathcal{K})$ be such that $\pi T' = \tau' \chi$. Then, writing χ as the $n \times n$ matrix (χ_{ij}) in $C(X) \otimes \mathfrak{M}_n$, and representing T' as the $n \times n$ matrix (T'_{ij}) in $\mathcal{L}(\mathcal{K}) \otimes \mathfrak{M}_n$, we also have $\tilde{\tau}\chi_{ij} = \pi_{\mathcal{K}} T'_{ij}$, $1 \leq i, j \leq n$. Thus $(\pi T'_{ij})$ is an $n \times n$ matrix of commuting normal entries in $\mathfrak{Q}(\mathcal{K})$, and hence T' is an essentially n-normal operator in $\mathcal{L}(\mathcal{K} \otimes \mathbf{C}^n) = \mathcal{L}(\mathcal{K}) \otimes \mathfrak{M}_n$. Let $\mathfrak{B} = C^*(\{\pi T'_{ij}\})$. Then $\pi T'$ is an element of $\mathfrak{B} \otimes \mathfrak{M}_n$, and from Lemma 2.3 $\Gamma_{\pi T'}$ is a $*$-isomorphism from $C^*(\pi T')$ onto $C^*_{\chi}(R^n(\pi T'))$. Notice that $R^n(\pi T') = R_e^n(T')$ and that the hypoconvex hull of the joint essential spectrum of T'_{ij} coincides with $R_e^n(T')$ (cf. Lemma 2.3). Let $\rho: C^*(\pi T') \to C^*_{\chi}(X)$ be the restriction of $(\tau')^{-1}$ to $C^*(\pi T')$ and let $L \in X$. Then the map $A \to \rho A(L)$, $A \in C^*(\pi T') \, (= C^*(\tau' \chi))$ defines a homomorphism in $\mathrm{Hom}(C^*(\pi T'), \mathfrak{M}_n)$ whose value at $\pi T'$ is L, and hence $L \in R_e^n(T')$. Therefore $X \subset R_e^n(T')$ and since $C^*_{\chi}(X)$ and $C^*_{\chi}(R_e^n(T'))$ are isomorphic, from Lemma 2.5 it follows that $\hat{X} = R_e^n(T')$. Furthermore, from the fact that $\tau' \sim \tau$, and Theorem 1.14, we deduce that if T is any operator in $\mathcal{L}(\mathcal{K})$ such that $\pi T = \tau \chi$, then $T \sim T'$, so that T is essentially n-normal and $R_e^n(T)$ $(= R_e^n(T')) = \hat{X}$. In order to prove the second part of the theorem, suppose that X is hypoconvex and that T is any essentially n-normal operator in $\mathcal{L}(\mathcal{K})$ such that $R_e^n(T) = X$. Also let $U: \mathcal{K} \to \mathcal{K} \otimes \mathbf{C}^n$ be a unitary transformation such that $T' = UTU^*$ can be identified with an $n \times n$ matrix in $\mathcal{L}(\mathcal{K}) \otimes \mathfrak{M}_n$ whose entries T'_{ij} are essentially commuting, essentially normal operators in $\mathcal{L}(\mathcal{K})$. Notice that $R_e^n(T) = R_e^n(T')$. It follows that $C^*(\pi T')$ is a C^*-subalgebra of $C^*(\{\pi T'_{ij}\}) \otimes \mathfrak{M}_n$. Let $Y \subseteq \mathfrak{M}_n$ be the joint essential spectrum of T'_{ij}, $1 \leq i, j \leq n$. Then, via the Gelfand transform(cf. Lemma 2.3), $C(Y)$ is $*$-isomorphic with $C(\{T'_{ij}\})$ and hence $\Gamma_{\pi T'}^{-1}$ extends to a $*$-isomorphism $\rho: C(Y) \otimes \mathfrak{M}_n \to C(\{\pi T'_{ij}\}) \otimes \mathfrak{M}_n$. We know, from Lemma 2.3, that $Y \subset X$ and $\hat{Y} = X$, so

that the inclusion map $Y \to X$ induces a homomorphism θ: $C(X) \otimes \mathfrak{M}_n \to C(Y) \otimes \mathfrak{M}_n$ whose restriction to $C_\chi^*(X)$, which we denote by $\mathfrak{T}$, is a $*$-isomorphism onto $C_\chi^*(Y)$ such that $\mathfrak{T}\chi = \chi$. Given a trivial extension τ_0 in $E_0(C(X) \otimes \mathfrak{M}_n)$, let τ_1 be the restriction of τ_0 to $C_\chi^*(X)$, and let $\tau' = \rho \circ \theta \oplus \tau_0$. Then $\tau' \in E(C(X) \otimes \mathfrak{M}_n)$ and $\tau'/C_\chi^*(X) = \rho \circ \mathfrak{T} \oplus \tau_1$. Since $\rho \circ \mathfrak{T} \oplus \tau_1 \sim_s \rho \circ \mathfrak{T}$, there exists a unitary V in $\mathfrak{L}(\mathfrak{K})$ such that $\pi V^*(\rho \circ \mathfrak{T}\chi \oplus \tau_1\chi)\pi V = \rho \circ \mathfrak{T}\chi = \pi T'$, so that if we define τ in $E(C(X) \otimes \mathfrak{M}_n)$ by $\tau A = \pi U^* V^* \tau' A \pi V U$ we obtain an extension of $C(X) \otimes \mathfrak{M}_n$ with the required properties.

Employing arguments similar to those given in the above proof we can obtain the following corollary, which gives an alternative proof of Corollary 1.12.

COROLLARY 3.6. *Let X be a hypoconvex subset of $\mathfrak{M}_n$. Then there exists a faithful unital representation ρ: $C_\chi^*(X) \to \mathfrak{L}(\mathfrak{K})$, such that $\rho[C_\chi^*(X)] \cap \mathfrak{K}(\mathfrak{K}) = \{0\}$, $\rho\chi$ is n-normal and $R''(\rho\chi) = R_e''(\rho\chi) = X$. Indeed, ρ can be taken to be the restriction to $C_\chi^*(X)$ of any faithful unital representation σ: $C(X) \otimes \mathfrak{M}_n \to \mathfrak{L}(\mathfrak{K})$, such that $\sigma(C(X) \otimes \mathfrak{M}_n) \cap \mathfrak{K}(\mathfrak{K}) = \{0\}$. On the other hand, if S is an n-normal operator in $\mathfrak{L}(\mathfrak{K})$ such that $R''(S) = R_e''(S) = X$, then there exists a unital faithful representation σ: $C(S) \otimes \mathfrak{M}_n \to \mathfrak{L}(\mathfrak{K})$ such that $\sigma(C(X) \otimes \mathfrak{M}_n) \cap \mathfrak{K}(\mathfrak{K}) = \{0\}$, and $\sigma\chi = S$.*

REMARK 3.7. Given a hypoconvex subset X of $\mathfrak{M}_n$, let Ξ_X: $\mathrm{Ext}(C(X) \otimes \mathfrak{M}_n) \to \mathrm{Ext}(C_\chi^*(X))$, and Ξ_X^s: $\mathrm{Ext}^s(C(X) \otimes \mathfrak{M}_n) \to \mathrm{Ext}^s(C_\chi^*(X))$ be the maps induced by restricting the elements of $E(C(X) \otimes \mathfrak{M}_n)$ to the C^*-subalgebra $C_\chi^*(X)$. In view of Lemma 3.5 it follows that $[\tau] \in$ range Ξ_X if and only if there exists $[T]$ in $EN_n(X)$ such that $\pi T = \tau\chi$ (a similar statement holds for $EN_n^s(X)$ and Ξ_X^s). Let ν: range $\Xi_X \to EN_n(X)$, μ: range $\Xi_X^s \to EN_n^s(X)$ be the maps defined by $\nu[\tau] = [T]$, $\mu[\tau]_s = [T]_s$ where $\pi T = \tau\chi$. From the above observation, it follows that ν and μ are well-defined isomorphisms. Following [30], we denote by $\mathrm{Ext}_\chi^n(X)$, the subgroup range Ξ_X of $\mathrm{Ext}(C_\chi^*(X))$.

The original proof of the following theorem, which was announced in [30], was based on the dilation method of [2] and the main result of [1]. Presently, it is an immediate consequence of the above remark and the fact that $\mathrm{Ext}(C(X) \otimes \mathfrak{M}_n)$ and $\mathrm{Ext}^s(C(X) \otimes \mathfrak{M}_n)$ are groups [see [11]].

THEOREM 3.8. *Let X be a hypoconvex subset of $\mathfrak{M}_n$. Then $EN_n(X)$ and $EN_n^s(S)$ are groups. Furthermore, the map $\beta_X = \nu \circ \Xi_X \circ \psi$ (ψ is as in Lemma 3.4) is a surjective homomorphism from $\mathrm{Ext}(X)$ to $EN_n(X)$.*

The following theorem can be used to effectively compute $EN_n(X)$ for an important class of hypoconvex sets.

THEOREM 3.9. *Let X be a hypoconvex subset of $\mathcal{G}\mathfrak{M}_n$. If X has a generating subset Y, then there exists a natural isomorphism λ: $\mathrm{Ext}(Y) \to EN_n(X)$, and the kernel of the map θ: $EN_n^s(X) \to EN_n(X)$ coincides with $\mathbf{Z}_n$.*

PROOF. We first show that range $\Xi_X = \mathrm{Ext}(C_X^*(X))$ and range $\Xi_X^s = \mathrm{Ext}^s(C_X^*(X))$. Since $\hat{Y} = X$, as in the proof of Lemma 3.5, the inclusion map $Y \to X$ induces a *-homomorphism θ: $C(X) \otimes \mathfrak{M}_n \to C(Y) \otimes \mathfrak{M}_n$ whose restriction to $C_X^*(X)$, which we denote by ζ, is a *-isomorphism onto $C_X^*(Y)$. From Theorem 2.9 we know that $C_X^*(Y) = C(Y) \otimes \mathfrak{M}_n$ and we deduce that the following diagrams are commutative:

$$
\begin{array}{ccc}
\mathrm{Ext}(C(X) \otimes \mathfrak{M}_n) & \overset{\Xi_X}{\to} & \mathrm{Ext}(C_X^*(X)) \\
\uparrow \theta_* & & \uparrow \zeta_* \\
\mathrm{Ext}(C(Y) \otimes \mathfrak{M}_n) & \underset{\mathrm{id}_*}{\to} & \mathrm{Ext}(C_X^*(Y))
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Ext}^s(C(X) \otimes \mathfrak{M}_n) & \overset{\Xi_X^s}{\to} & \mathrm{Ext}^s(C_X^*(X)) \\
\uparrow \theta_*^s & & \uparrow \zeta_*^s \\
\mathrm{Ext}^s(C(Y) \otimes \mathfrak{M}_n) & \underset{\mathrm{id}_*^s}{\to} & \mathrm{Ext}^s(C_X^*(X))
\end{array}
$$

Since ζ_* and ζ_*^s are isomorphisms, the map Ξ_X and Ξ_X^s are onto and our claim is established. In particular, the map ν (introduced in Remark 3.7) is an isomorphism from $\mathrm{Ext}(C_X^*(X)) = \mathrm{Ext}_X^n(X)$ onto $EN_n(X)$. Let $\lambda = \nu \circ \zeta_* \circ \Psi$, where Ψ is as in Lemma 3.4. Then λ is an isomorphism from $\mathrm{Ext}(Y)$ onto $EN_n(X)$. Now the proof of the theorem is completed by using Lemma 3.3, and observing that the following diagram is commutative:

$$
\begin{array}{ccccc}
\mathrm{Ext}^s(C(Y) \otimes \mathfrak{M}_n) & \overset{\zeta_*^s}{\to} & \mathrm{Ext}^s C_X^*(X) & \overset{\mu}{\to} & EN_n^s(X) \\
\downarrow \Phi & & \downarrow \Phi & & \downarrow \theta \\
\mathrm{Ext}(C(Y) \otimes \mathfrak{M}_n) & \overset{\zeta_*}{\to} & \mathrm{Ext}\, C_X^*(X) & \overset{\nu}{\to} & EN_n(X)
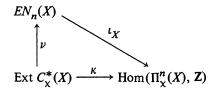\end{array}
$$

REMARK 3.10. The above theorem allows us to apply the machinery developed in [11] and [22]. In particular, if Y is homotopically equivalent to a subset of the plane, then $EN_n(X)$ is isomorphic to the first cohomology group of Y, and if Y is homotopically equivalent to a subset of $\mathbf{C}^2$, then $EN_n(X)$ is isomorphic to the direct sum of the first Steenrod homology group of Y with the third Steenrod homology group of Y.

DEFINITION 3.11. Given T in $\mathcal{L}(\mathcal{H})$ suppose that $\bigcap_{\varphi \in \mathrm{Hom}(C^*(\pi T), \mathfrak{M}_n)} \ker \varphi = \{0\}$. For each $f \in C_X^*[R_\epsilon^n(T)]$ we define $f(\pi T) = \Gamma_{\pi T}^{-1} f$, where $\Gamma_{\pi T}$ is defined as in Definition 2.1 (note that we then have the spectral mapping formula: $R^n[f(\pi T)] = (f[R_\epsilon^n(T)])^\hat{}$). Also, given a nonempty compact subset X of $\mathfrak{M}_n$ we denote by $\Pi_X^n(X)$ the set of all homotopy classes of invertible

elements in $C_X^*(X)$ (notice that $\Pi_X^n(X)$ has a natural group structure since it is isomorphic to the index group of $C_X^*(X)$ [**19**, Chapter 2]). Furthermore, let $\iota_X \colon EN_n(X) \to \mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})$ be the homomorphism given by $(\iota_X[T])[f] = \mathrm{ind}\, f(\pi T)$, for every $[f] \in \Pi_X^n(X)$, $[T] \in EN_n(X)$ [**30**].

THEOREM 3.12. *Let X be a hypoconvex subset of $\mathcal{G}\mathfrak{M}_n$ and suppose that X has a generating subset Y which is homotopically equivalent to a compact subset of* **C.** *Then, ι_X is an isomorphism. In particular, if $\{[f_j]; j = 1, 2, \ldots \}$ is a system of generators for $\Pi_X^n(X)$, then two essentially n-normal operators T_1 and T_2 in $\mathcal{L}(\mathcal{K})$ such that $R_e^n(T_j) = X$ are weakly equivalent if and only if $\mathrm{ind}\, f_j(\pi T_1) = \mathrm{ind}\, f_j(\pi T_2)$ for $j = 1, 2, \ldots$.*

PROOF. As in Theorem 3.9 we observe that $\mathrm{Ext}\, C_X^*(X) = \mathrm{Ext}_X^n(X)$, so that ν is an isomorphism from $\mathrm{Ext}\, C_X^*(X) \to EN_n(X)$ and if we define the homomorphism $\kappa \colon \mathrm{Ext}\, C_X^*(X) \to \mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})$ by $\kappa[\tau][f] = \mathrm{ind}\,\tau f$ where f is an invertible element of $C_X^*(X)$, then the following diagram is easily seen to be commutative:

$$
\begin{array}{ccc}
EN_n(X) & & \\
\uparrow{\scriptstyle\nu} & \searrow{\scriptstyle \iota_X} & \\
\mathrm{Ext}\, C_X^*(X) & \xrightarrow{\ \kappa\ } & \mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})
\end{array}
$$

Again, as in the proof of Theorem 3.9, we let $\zeta \colon C_X^*(X) \to C_X^*(Y)\ (= C(Y) \otimes \mathfrak{M}_n)$ be the $*$-isomorphism induced by the inclusion $Y \to X$. Therefore, ζ induces an isomorphism $\bar{\zeta} \colon \Pi_X^n(X) \to \Pi_X^n(Y)$ so that, by duality, we have an isomorphism $\bar{\zeta}^* \colon \mathrm{Hom}(\Pi_X^n(Y), \mathbf{Z}) \to \mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})$. As $\Pi_X^n(Y)$ is the index group of $C(Y) \otimes \mathfrak{M}_n$, it can be naturally identified with the set $[Y, \mathfrak{A}_u]$ of homotopy classes from Y into the unitary group $\mathfrak{A}_n$ in $\mathfrak{M}_n$. Since Y is homotopically equivalent to a subset of the plane, we can use a standard argument with the homotopy sequence of the fibration $\mathfrak{A}_k \to \mathfrak{A}_{k+1} \to S^{2k+1}$ (where S^{2k+1} is the $2k + 1$st sphere) (see [**32**, Chapter 7, §2], [**4**, p. 239], [**6**, §1]), to show that the map $j \colon [Y, \mathfrak{A}_1]\ (= \Pi^1(Y)) \to [Y, \mathfrak{A}_n]$ given by

$$
j[f] = \left[\begin{pmatrix} f & 0 \\ 0 & 1_{n-1} \end{pmatrix} \right],
$$

for $[f] \in [Y, \mathfrak{A}_1]$ is an isomorphism. The proof of the theorem is completed by observing that the following diagram is commutative:

$$
\begin{array}{ccccc}
\mathrm{Ext}(Y) & \xrightarrow{\ \Psi\ } & \mathrm{Ext}(C(Y) \otimes \mathfrak{M}_n) & \xrightarrow{\ \zeta_*\ } & \mathrm{Ext}\, C_X^*(X) \\
\downarrow{\scriptstyle\gamma_Y} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\
\mathrm{Hom}(\Pi^1(Y), \mathbf{Z}) & \xleftarrow[j^*]{} & \mathrm{Hom}(\Pi_X^n(Y), \mathbf{Z}) & \xrightarrow[\bar{\zeta}^*]{} & \mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})
\end{array}
$$

where γ_Y is the isomorphism of [**10**, Theorem 10.5].

REMARK 3.13. Even when X is a hypoconvex subset of $\mathfrak{T}\mathfrak{M}_2$, the map ι_X may not be injective. Indeed, let Z be a compact subset of $\mathbf{C}^2$ homeomorphic to the real projective plane, such that $|\lambda_1| > |\lambda_2|$ for every $(\lambda_1, \lambda_2) \in Z$. Let $Y \subset \mathfrak{M}_2$ defined by

$$Y = \left\{ \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} : (\lambda_1, \lambda_2) \in Z \right\}.$$

Then any two different matrices of Y are inequivalent and irreducible and hence Y is a generating set for $X = \hat{Y}$. From Theorem 3.9 and [19], it follows that $EN_2(X) \simeq \mathrm{Ext}(Y) \simeq \mathbf{Z}_2$, and hence kernel $\iota_X = EN_2(X) = \mathbf{Z}_2 \neq \{0\}$ (notice that $\mathrm{Hom}(\Pi_X^n(X), \mathbf{Z})$ is torsion free).

Our next objective is to prove the homotopy invariance of $EN_n(\cdot)$ and $EN_n^\cdot(\cdot)$. We start with the following lemma whose proof is elementary and will be omitted.

LEMMA 3.14. *Let* $X, Y \in \mathbf{HC}_n$. θ: $[0, 1] \to \mathrm{Hom}(C_X^*(Y), C_X^*(X))$ *is a continuous function, and if we define* h: $[0, 1] \to \mathrm{Hom}(X, Y)$ *by* $h(t) = [\theta(t)]_X$, *then* h *is a continuous function, where* $\mathrm{Hom}(X, Y)$ *is endowed with the topology of uniform convergence (i.e. the topology inherited by that of* $C_X^*(X)$). *Conversely, if* h: $[0, 1] \to \mathrm{Hom}(X, Y)$ *is a continuous function, then the function* θ: $[0, 1] \to$ $\mathrm{Hom}(C_X^*(Y), C_X^*(X))$ *defined by* $\theta(t) = h(t)^*$, $t \in [0, 1]$, *is also continuous.*

DEFINITION 3.15. Given $f \in \mathrm{Hom}(X, Y)$ we denote by $f_*^\cdot$: $EN_n^\cdot(X) \to EN_n^\cdot(Y)$, and by f_*: $EN_n(X) \to EN_n(Y)$, the maps given by $f_*^\cdot[T]_s = [S_f]_s$ and $f_*[T] = [S_f]$, where S_f is any operator in $\mathcal{L}(\mathcal{K})$ such that $\pi S_f = f(\pi T)$.

REMARK 3.16. It is easy to check that the maps $f_*^\cdot$ and f_* are well defined. On the other hand, since for each $f \in \mathrm{Hom}(X, Y)$, $X, Y \in \mathbf{HC}_n$, the following diagrams commute:

$$
\begin{array}{ccc}
\mathrm{Ext}^s(C(X) \otimes \mathfrak{M}_n) & \xrightarrow{\;\Xi_X^s\;} & \mathrm{Ext}^s(C_X^*(X)) \\
(f^*)_*^s \downarrow & & \downarrow (f^*)_*^s \\
\mathrm{Ext}^s(C(Y) \otimes \mathfrak{M}_n) & \xrightarrow[\;\Xi_Y^s\;]{} & \mathrm{Ext}^s(C_X^*(Y))
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Ext}(C(X) \otimes \mathfrak{M}_n) & \xrightarrow{\;\Xi_X\;} & \mathrm{Ext}(C_X^*(X)) \\
(f^*)_* \downarrow & & \downarrow (f^*)_* \\
\mathrm{Ext}(C(Y) \otimes \mathfrak{M}_n) & \xrightarrow[\;\Xi_X\;]{} & \mathrm{Ext}(C_X^*(Y))
\end{array}
$$

we deduce that the following diagrams are also commutative:

$$
\begin{array}{ccc}
\text{range}\,\Xi^s_X & \xrightarrow{\ \mu\ } & EN^s_n(X) \\
\downarrow{\scriptstyle (f^*)^s_*} & & \downarrow{\scriptstyle f^s_*} \\
\text{range}\,\Xi^s_Y & \xrightarrow{\ \mu\ } & EN^s_n(Y)
\end{array}
\qquad
\begin{array}{ccc}
\text{range}\,\Xi_X & \xrightarrow{\ \nu\ } & EN_n(X) \\
\downarrow{\scriptstyle (f^*)_*} & & \downarrow{\scriptstyle f_*} \\
\text{range}\,\Xi_Y & \xrightarrow{\ \nu\ } & EN_n(X)
\end{array}
$$

The above observations show that $EN^s_n(\cdot)$ and $EN_n(\cdot)$ have functorial properties. Indeed, we have the following theorem.

THEOREM 3.17. $EN^s_n(\cdot)$ and $EN_n(\cdot)$ are homotopy invariant covariant functors from the category $\mathbf{HC}_n$ into the category of abelian groups. In particular, if $X, Y \in \mathbf{HC}_n$, and $h\colon [0, 1] \to \mathrm{Hom}(X, Y)$ is a continuous function, then $h(0)^s_* = h(1)^s_*$, and $h(0)_* = h(1)_*$.

PROOF. We only prove the theorem for the functor $EN^s_n(\cdot)$ since the proof for $EN_n(\cdot)$ is similar. For X, Y, h as in the statement of the theorem and for $t \in [0, 1]$, we see that the following diagrams are commutative:

$$
\begin{array}{ccc}
\mathrm{Ext}^s(C(X)\otimes \mathfrak{M}_n) & \xrightarrow{\ \Xi^s_X\ } & \mathrm{Ext}^s(C^*_X(X)) \\
\downarrow{\scriptstyle (h(t)^*)^s_*} & & \downarrow{\scriptstyle (h(t)^*)^s_*} \\
\mathrm{Ext}^s(C(Y)\otimes \mathfrak{M}_n) & \xrightarrow{\ \Xi^s_X\ } & \mathrm{Ext}^s(C^*_X(Y))
\end{array}
$$

$$
\begin{array}{ccc}
\text{range}\,\Xi^s_X & \xrightarrow{\ \mu\ } & EN^s_n(X) \\
\downarrow{\scriptstyle (h(t)^*)^s_*} & & \downarrow{\scriptstyle h(t)^s_*} \\
\text{range}\,\Xi^s_Y & \xrightarrow{\ \mu\ } & EN^s_n(Y)
\end{array}
$$

From Theorem 3.1 and Lemma 3.14 we conclude that $(h(0)^*)^s_* = (h(1)^*)^s_*$ and hence that $h(0)^s_* = h(1)^s_*$, as desired.

COROLLARY 3.18. Let $\mathfrak{B}_n$ be the unit ball in $\mathfrak{M}_n$, then $EN^s_n(\mathfrak{B}_n) = EN_n(\mathfrak{B}_n) = \{0\}$. In particular, if T is an essentially n-normal operator in $\mathcal{L}(\mathcal{H})$ such that $R^n_e(T) = \mathfrak{B}_n$, then T is a compact perturbation of n-normal operator S in $\mathcal{L}(\mathcal{H})$ such that $R^n_e(S) = \mathfrak{B}_n$.

PROOF. Let $h: [0, 1] \to \operatorname{Hom}(\mathfrak{B}_n, \mathfrak{B}_n)$ be given by $h(t) = t\chi$, where $t \in [0, 1]$ and χ is the identity function on $\mathfrak{B}_n$. Then $h(1)^\zeta_*$ is the identity on $EN_n^\zeta(\mathfrak{B}_n)$ while $h(0)^\zeta_*$ is the zero homomorphism.

Our final task in this section is to present a partial answer to a question raised in [30].

REMARK 3.19. It is well known that if every irreducible representation of $C^*(T)$ is at most n-dimensional, then the operator T can be expressed as $T = \sum_{k=1}^n \oplus T_k$, where T_k is a k-normal and some summands might be missing (cf. [12]). In particular, if $C^*(T)$ is n-homogeneous, i.e. if every irreducible representation of $C^*(T)$ is n-dimensional, then T must be n-normal. For $n > 2$, it is not known (and likely to be false) whether the Calkin algebra version of the last fact holds. The following theorem shows that for $n \leqslant 2$ this question has a positive answer.

THEOREM 3.20. *If every irreducible representation $\pi(C^*(T))$ is 2-dimensional, then the operator T is essentially binormal.*

PROOF. It readily follows that if every irreducible representation of $\pi(C^*(T))$ is at most 2-dimensional, then the map $\Gamma_{\pi T}: C^*(\pi T) \to C_X^*(R^2(\pi T))$ (cf. Definition 2.1) is an isomorphism. Now let $\tau = (\Gamma_{\pi T})^{-1}$, and let $X = R^2(\pi T)$ $(= R_e^2(T))$. Since, by hypothesis, every irreducible representation of $\pi(C^*(T))$ (and hence of $C_X^*(X)$) is 2 dimensional, we deduce that X is a hypoconvex subset of $\mathcal{I}\mathfrak{M}_2$. From Remark 2.10 we then see that there exists a generating set for X. As we remarked previously, the inclusion map $Y \to X$ induces a $*$-isomorphism $\zeta: C_X^*(X) \to C_X^*(Y)$ and a $*$-homomorphism $\theta: C(X) \otimes \mathfrak{M}_2 \to C(Y) \otimes \mathfrak{M}_2$. Therefore, the following diagram is commutative:

$$
\begin{array}{ccc}
C(Y) \times \mathfrak{M}_2 & \overset{\theta_*}{\to} & C(X) \times \mathfrak{M}_2 \\
(\mathrm{id})_* \downarrow & & \downarrow \Xi_X \\
C_X^*(Y) & \underset{\zeta_*}{\to} & C_X^*(X)
\end{array}
$$

Since $(\mathrm{id})_*$ and ζ_* are isomorphisms, it follows that Ξ_X is surjective. In particular, if τ is as above, then $[\tau] \in$ range of Ξ_X, and hence T is essentially binormal.

4. Toeplitz operators on strongly pseudoconvex domains. In this section we apply the results of preceding ones to obtain a homotopic property of Toeplitz operators with matricial continuous symbols on strongly pseudoconvex domains. We start by proving some few facts concerning the C^*-algebra generated by the set of Toeplitz operators. Our presentation follows the pattern of [15].

In what follows, Ω will denote a strongly pseudoconvex domain (also called strictly Levi-pseudoconvex, cf. [20]) in $\mathbf{C}^m$ with smooth boundary. $L^2(\Omega)$ will denote the Hilbert space of all complex valued square integrable functions on Ω with respect to Lebesgue measure, and $H^2(\Omega)$ will denote the subspace of all holomorphic functions in $L^2(\Omega)$. We shall denote by $P: L^2(\Omega) \to H^2(\Omega)$ the (orthogonal) projection onto $H^2(\Omega)$, and $k(z, w)$, $(z, w) \in \Omega \times \Omega$ (the Bergman kernel [5]) will denote the reproducing kernel of $H^2(\Omega)$. We recall that for each $w \in \Omega$, $k(\cdot, w) \in H^2(\Omega)$ and for every $f \in L^2(\Omega)$, $Pf(z) = \int_\Omega k(z, w)f(w)dw$. In particular, $k(z, w) = \sum_{j=1}^{\infty} g_j(z)\bar{g}_j(w)$, where $\{g_j, j = 1, 2, \ldots\}$ is an orthonormal basis for $H^2(\Omega)$. From [23] it follows that $k(z, w)$ is a C^{∞} function up to the boundary of $\Omega \times \Omega$, except for the diagonal $\{(z, w) \in \partial\Omega \times \partial\Omega: z = w\}$. Also, for $z \in \Omega$, $k(z, z) > 0$, and $\lim_{z \to z_0} k(z, z) = \infty$ for every $z_0 \in \partial\Omega$ (see [7]).

DEFINITION 4.1 (SEE [33]). Given $\varphi \in C(\overline{\Omega})$, we denote by M_φ the operator on $L^2(\Omega)$ defined by multiplication by φ, and by T_φ the operator on $H^2(\Omega)$ defined by $T_\varphi f = P\varphi f = PM_\varphi f$, $f \in H^2(\Omega)$. Also, the norm closed subalgebra of $\mathcal{L}(L^2(\Omega))$ generated by all the operators T_φ, $\varphi \in C(\overline{\Omega})$, will be denoted by $\mathfrak{T}(\Omega)$.

LEMMA 4.2. *Let* $\varphi \in C(\overline{\Omega})$.
(a) $PM_\varphi - M_\varphi P$ *is compact on* $L^2(\Omega)$.
(b) T_φ *is compact on* $H^2(\Omega)$ *if and only if* $\varphi|\partial\Omega = 0$.

PROOF. Statement (a) was proved in [33, Theorem 2.1] for the case that $\varphi \in C^{\infty}(\Omega)$. Since these functions are dense in $C(\overline{\Omega})$ and $\mathcal{K}(L^2(\Omega))$ is closed, the validity of part (a) follows. Furthermore, in [33, Theorem 2.3] it was shown that if $\varphi|\partial\Omega = 0$ and $\varphi \in C^{\infty}(\Omega)$, then T_φ is compact. Thus one half of part (b) follows using the same argument employed previously. For the other half of part (b), suppose that T_φ is compact for some $\varphi \in C(\overline{\Omega})$. Let z_0 be an arbitrary point in $\partial\Omega$, let $\{z_j\}$ be a sequence in Ω that converges to z_0, and let f_j be the function in $H^2(\Omega)$ defined by $f_j(z) = k(z, z_j)/[k(z_j, z_j)]^{1/2}$, $j = 1, 2, \ldots$, $z \in \Omega$. It readily follows, using the fact that k is a reproducing kernel, that $\|f_j\| = 1$ for $j = 1, 2, \ldots$. Also, from [23] we see that for every open neighborhood ω of z_0, $f_j \in C^{\infty}(\overline{\Omega} \setminus \omega)$ and converges uniformly to zero on $\overline{\Omega} \setminus \omega$. Therefore,

$$\lim_{j \to \infty} \|[T_\varphi - \varphi(z_0)]f_j\|^2 \leqslant \lim_{j \to \infty} \int_\Omega |\varphi(w) - \varphi(z_0)|^2 |f_j(w)|^2 dw = 0.$$

On the other hand, using again the fact that k is a reproducing kernel, we see that for every bounded $f \in H^2(\Omega)$,

$$\lim_{j \to \infty} \langle f, f_j \rangle = \lim_{j \to \infty} f(z_j)/[k(z_j, z_j)]^{1/2} = 0$$

and hence $\lim_{j \to \infty} \langle f, f_j \rangle = 0$ for every $f \in H^2(\Omega)$. Here we have used the

standard fact that $H^\infty(\Omega)$ is dense in $H^2(\Omega)$. Thus, $\{f_j\}$ converges weakly to zero, and since T_φ is compact, we see that $\lim_{j\to\infty}\|T_\varphi f_j\| = 0$. We conclude that

$$|\varphi(z_0)| \leqslant \lim_{j\to\infty} \left\{ \|\left[T_\varphi - \varphi(z_0)\right]f_j\| + \|T_\varphi f_j\| \right\} = 0$$

as desired.

REMARK 4.3. Reasoning just as in [15, Lemma 1] it follows that $\mathfrak{T}(\Omega)$ is an irreducible C^*-algebra on $H^2(\Omega)$, and since it clearly contains nonzero compact operators, we see that $\mathfrak{K}[H^2(\Omega)] \subset \mathfrak{T}(\Omega)$. Let $\varphi \in C(\partial\Omega)$ and let $\varphi^0 \in C(\overline{\Omega})$ be any continuous extension of φ. It follows from Lemma 4.2 that the element πT_{φ^0} in $\mathfrak{Q}[H^2(\Omega)]$ is independent of the choice of the extension φ^0 of φ.

LEMMA 4.4. *Let* $\tau: C(\partial\Omega) \to \mathfrak{Q}[H^2(\Omega)]$ *be defined by* $\tau\varphi = \pi T_{\varphi^0}$, $\varphi \in C(\partial\Omega)$. *Then* $\tau \in E[C(\partial\Omega)]$ (*i.e.*, τ *is a unital $*$-monomorphism*).

PROOF. In view of Lemma 4.2(b) we need only check that τ is a homomorphism. Let $\varphi, \psi \in C(\partial\Omega)$. Then,

$$(\tau\varphi)(\tau\psi) = (\pi T_{\varphi^0})(\pi T_{\psi^0}) = \pi T_{\varphi^0} T_{\psi^0}$$
$$= \pi\left(T_{\varphi^0\psi^0} - PM_{\varphi^0}(1 - P)M_{\psi^0}|H^2(\Omega)\right) = \pi T_{\varphi^0\psi^0} = \tau(\varphi\psi),$$

where we have used Lemma 4.2(a).

DEFINITION 4.5. Let $\mathfrak{K} = H^2(\Omega) \otimes \mathbf{C}^n$ and let $P_n = P \otimes 1_n$, where 1_n is the identity matrix in $\mathfrak{M}_n$. Given $\varphi \in C(\overline{\Omega}) \otimes \mathfrak{M}_n$ define T_φ on $\mathfrak{K}$ by $T_\varphi x = P_n\varphi x$, $x \in \mathfrak{K}$. Also, let $\mathfrak{T}_n(\Omega)$ be the norm closed subalgebra of $\mathcal{L}(\mathfrak{K})$ generated by T_φ, for $\varphi \in C(\partial\Omega) \otimes \mathfrak{M}_n$.

REMARK 4.6. An easy verification shows that $\mathfrak{T}_n(\Omega)$ can be identified with $\mathfrak{T}(\Omega) \otimes \mathfrak{M}_n$. On the other hand, from Lemma 4.4, it follows that the sequence

$$0 \to \mathfrak{K}\left[H^2(\Omega)\right] \to \mathfrak{T}(\Omega) \to C(\partial\Omega) \to 0$$

is exact. Since tensoring by $\mathfrak{M}_n$ preserves exactness and $\mathfrak{K}[H^2(\Omega)] \otimes \mathfrak{M}_n$ can be naturally identified with $\mathfrak{K}(\mathfrak{K})$, we deduce that the following sequence is also exact

$$0 \to \mathfrak{K}(\mathfrak{K}) \to \tau_n(\Omega) \to C(\partial\Omega) \otimes \mathfrak{M}_n \to 0.$$

As in Remark 4.3, given $\varphi \in C(\partial\Omega) \otimes \mathfrak{M}_n$ we denote by φ^0 any continuous extension of φ to $\overline{\Omega}$.

The following corollary is a consequence of Lemma 4.4 and the above remark.

COROLLARY 4.7. *Let* $\tau_n: C(\partial\Omega) \otimes \mathfrak{M}_n \to \mathfrak{Q}(\mathfrak{K})$ *given by* $\tau_n\varphi = \pi T_{\varphi^0}$, $\varphi \in C(\partial\Omega) \otimes \mathfrak{M}_n$. *Then* $\tau_n \in E[C(\partial\Omega) \otimes \mathfrak{M}_n]$. *In fact*, $\tau_n = \tau \otimes 1_n$, *where* τ *is as in Lemma 4.4.*

With the above result we are finished with the preliminary material needed for the application of the previous sections.

THEOREM 4.8. *Let* $\varphi, \psi \in C(\partial\Omega) \otimes \mathfrak{M}_n$ *and suppose that* $X \in \mathbf{HC}_n$ *and that* range $\varphi \subset X$, range $\psi \subset X$. *Furthermore, assume that there exist a continuous function* $h\colon [0, 1] \to C(\partial\Omega) \otimes \mathfrak{M}_n$, $h(0) = \varphi$, $h(1) = \psi$, *and* range $h(t) \subset X$ *for every* $0 \leqslant t \leqslant 1$. *Then, for every* n-*normal operator* S *in* $\mathfrak{L}(\mathfrak{K})$ *such that* $R_e^n(S) = X$, *we have* $T_{\varphi^\circ} \oplus S \sim_s T_{\psi^\circ} \oplus S$. *Moreover, if* [range φ]$^\wedge$ = [range ψ]$^\wedge$ = X, *it follows that* $T_{\varphi^\circ} \sim_s T_{\psi^\circ}$, *i.e. there exists a unitary* U *in* $\mathfrak{L}(\mathfrak{K})$ *such that* $U^* T_{\varphi^\circ} U - T_{\psi^\circ} \in \mathfrak{K}(\mathfrak{K})$.

PROOF. Let $\varphi^*\colon C_\chi^*(X) \to C(\partial\Omega) \otimes \mathfrak{M}_n$ defined by $\varphi^* f = f \circ \varphi$, $f \in C_\chi^*(X)$. Furthermore, let $\theta\colon [0, 1] \to \mathrm{Hom}(C_\chi^*(X), C(\partial\Omega) \otimes \mathfrak{M}_n)$ be defined by $\theta(t) = h(t)^*$, $0 \leqslant t \leqslant 1$. Then it readily follows that θ is a continuous function, $\theta(0) = \varphi^*$ and $\theta(1) = \psi^*$. Since every irreducible representation of $C_\chi^*(X)$ and $C(\partial\Omega) \otimes \mathfrak{M}_n$ is at most n-dimensional (see [17, Chapter 3]), from Theorem 3.1 we deduce that $\theta(0)_*^\tau = \theta(1)_*^\tau$. In particular, $(\varphi^*)_*^\tau[\tau_n]_s = (\psi^*)_*^\tau[\tau_n]_s$. This means that for $\sigma \in E_0(C(X) \otimes \mathfrak{M}_n)$, there exists a unitary U in $\mathfrak{L}(\mathfrak{K})$ such that

$$\pi U^* [\tau_n(f \circ \varphi) \oplus \sigma f] \pi U = \tau_n(f \circ \psi) \oplus \sigma f, \quad \text{for every } f \in C_\chi^*(X). \quad (\sharp)$$

Since $\tau_n \in E[C(\partial\Omega) \otimes \mathfrak{M}_n]$, the restriction of τ_n to $C^*(\varphi)$ defines a $*$-isomorphism from $C^*(\varphi)$ onto $C^*(\pi T_{\varphi^\circ})$. From Corollary 2.4 it follows that $R_e^n(T_{\varphi^\circ}) = $ [range ψ]$^\wedge \subset X$. Let S in $\mathfrak{L}(\mathfrak{K})$ be an n-normal operator such that $\pi S = \sigma\chi$ (see Corollary 3.6). Then, it follows that $R_e^n(S) = X$ and, applying $(\sharp)$ for $f = \chi$, we obtain: $U^*(T_{\varphi^\circ} \oplus S)U - T_{\psi^\circ} \oplus S \in \mathfrak{K}(\mathfrak{K})$. Now, if [range φ]$^\wedge$ = [range ψ]$^\wedge$ = X, then, as noted previously, $R_e^n(T_{\varphi^\circ}) = R_e^n(T_{\psi^\circ}) = X$. Therefore, from the results mentioned in Remark 1.5 (see [27]) we conclude that $T_{\varphi^\circ} \sim_s T_{\varphi^\circ} \oplus S \sim_s T_{\psi^\circ}$, as required.

REMARK 4.9. (a) The converse of the last part of Theorem 4.8 is false in general (at least in lower dimensions). What is too strong is the homotopy condition, because the condition about the ranges of φ and ψ is a necessary one. Let X be a "figure 8", i.e.

$$X = \{\lambda \in \mathbf{C}\colon |\lambda - 1| = 1\} \cup \{\lambda \in \mathbf{C}\colon |\lambda + 1| = 1\}.$$

Let φ and ψ be the complex valued continuous functions on the unit circle S^1 defined by

$$\varphi(\lambda) = \begin{cases} 1 + \lambda^2, & |\arg \lambda| \leqslant \pi/2, \\ -1 - \lambda^2, & \pi/2 \leqslant |\arg \lambda| \leqslant \pi; \end{cases}$$

$$\psi(\lambda) = \begin{cases} -1 - \lambda^2, & |\arg \lambda| \leqslant \pi/2, \\ 1 + \lambda^2, & \pi/2 \leqslant |\arg \lambda| \leqslant \pi. \end{cases}$$

Then, if $\omega(f)$ denotes the winding number of an invertible function $f \in C(S^1)$ about the origin, we have $\omega(\varphi - 1) = \omega(\psi - 1) = \omega(\varphi + 1) = \omega(\psi + 1) = 1$. Since range φ = range $\psi = X$, and T_φ and T_ψ are essentially normal operators, it follows that the essential spectra of T_φ and T_ψ coincide with X, and $\mathrm{ind}(T_\varphi - \lambda) = \mathrm{ind}(T_\psi - \lambda)$, for every $\lambda \notin X$. Therefore, from [10, Theorem 11.1] $T_\varphi \sim_s T_\psi$. However, φ and ψ are not in the same homotopy class of the fundamental group $\pi_1(X)$, because this is a free group on two generators, and $[\varphi]$ and $[\psi]$ represent two different reduced words. It would be interesting to find out whether the phenomenon presented here is due to the noncommutativity of the fundamental group of X.

(b) Let φ and $\psi \in C(S^1) \otimes \mathfrak{M}_2$ given by $\varphi(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\psi(\lambda) = \begin{pmatrix} \lambda & 2 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in S^1$. T_{φ^o} and T_{ψ^o} are compact perturbations of $\begin{pmatrix} U^+ & 1 \\ 0 & U^* \end{pmatrix}$ and $\begin{pmatrix} U^+ & 2 \\ 0 & U^* \end{pmatrix}$, respectively, where U^+ is a simple unilateral shift. Let $Y \subset \mathfrak{M}_2$ be the set of all matrices of the form $Y = \{\begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}, \; |\lambda| = 1, \; 1 < \alpha < 2\}$. Then $[\text{range } \varphi]^\cap \subset \hat{Y}$ and $[\text{range } \psi]^\cap \subset \hat{Y}$. If S is any binormal operator such that $R_e^2(S) = \hat{Y}$ (which can be obtained by using Corollary 1.12), then, from Theorem 4.8, $\begin{pmatrix} U^+ & 1 \\ 0 & U^* \end{pmatrix} \oplus S$ is strongly equivalent to $\begin{pmatrix} U^+ & 2 \\ 0 & U^* \end{pmatrix} \oplus S$. The idea of this example was suggested to us by Vern Paulsen, who proved that $\begin{pmatrix} U^+ & 2 \\ 0 & U^* \end{pmatrix} \oplus S$ is weakly equivalent to $\begin{pmatrix} U^+ & 2 \\ 0 & U^* \end{pmatrix} \oplus S$ employing the invariant that he introduced in [25] and the classification of essentially normal operators discovered by Brown, Douglas and Fillmore [10, Theorem 11.1].

DEFINITION 4.10. Let X be a hypoconvex subset of $\mathfrak{M}_n$ and let $L \in X$. Let $\xi_m: \Pi_m(X, L) \to EN_n(X)$ be the map given by $\xi_m[\varphi] = [T_{\varphi^o} \oplus S]$, where S is some n-normal operator in $\mathcal{L}(\mathcal{K})$ such that $R_e^n(S) = X$. Hereafter, $\Pi_m(X, L)$ denotes the $(2m - 1)$th homotopy group of X with base point L [32, Chapter 7, §2]. In the following theorem we see that ξ_m resembles the Hurewicz homomorphism [32, Chapter 7, §4].

THEOREM 4.11. *Let B_m be the closed unit ball in $\mathbb{C}^m$. If $\tau_n \in E[C(\partial B_m) \otimes \mathfrak{M}_n]$, and ν: range $\Xi_X \to EN_n(X)$ are defined as in Corollary 4.7 and Remark 3.7, then $\xi_m[\varphi] = (\nu \circ (\varphi^*)_*)[\tau_n]$, for every $[\varphi] \in \Pi_m(X, L)$. Furthermore, ξ_m is a group homomorphism.*

PROOF. Since the first part of the theorem follows immediately from the definitions of the maps involved, we shall be concerned with the second half. Let $Z_1 = \partial B_m = Z_2$, let Z be the disjoint union of Z_1 and Z_2 with the base point identified, and let $g: \partial B_m \to Z$ be the map that collapses the equator of ∂B_m to the base point. Let $[\varphi_j] \in \Pi_m(X, L)$, define $\varphi: Z \to X$ such that $\varphi|Z_j = \varphi_j, j = 1, 2$. Then the product of $[\varphi_1]$ and $[\varphi_2]$ in $\Pi_m(X, L)$ is defined as $[\varphi_1] \circ [\varphi_2] = [\varphi \circ g]$ (recall that this operation is abelian for $m > 1$, cf. [32, Chapter 7, §2]). Let $i_j: Z_j \to Z$ be the canonical inclusion maps and let $\beta(a, b) = i_{1*}a + i_{2*}b$, $a \in \mathrm{Ext}(Z_1)$, $b \in \mathrm{Ext}(Z_2)$. From [10, Corollary 6.2] it follows that β is an isomorphism. A right inverse for β is easily seen to be the

map (r_{1*}, r_{2*}) where $r_j: Z \to Z_j$ is the retraction obtained by collapsing Z_{3-j} to the base point, $j = 1, 2$. Now notice that $r_j g: \partial B_m \to \partial B_m$ is homotopic to the identity map, so that $((r_{1*}, r_{2*}) \circ g_*)[\tau_n] = ((r_1 \circ g)_*, (r_2 \circ g)_*)[\tau_n] = ([\tau_n], [\tau_n])$. On the other hand,

$$(\varphi^*)_* \circ \beta = (\varphi^*)_* \circ (i_{1*} + i_{2*})$$

$$= ((\varphi \circ i_1)^*)_* + ((\varphi \circ i_2)^*)_* = (\varphi_1^*)_* + (\varphi_2^*)_*.$$

Therefore,

$$\xi_m([\varphi_1] \circ [\varphi_2]) = (\nu \circ ((\varphi \circ g)^*)_*)[\tau_n] = (\nu \circ (\varphi^*)_* \circ g_*)[\tau_n]$$

$$= (\nu \circ (\varphi^*)_* \beta(r_{1*}, r_{2*}) \circ g_*)[\tau_n]$$

$$= (\nu \circ (\varphi_1^*)_*)[\tau_n] + (\nu \circ (\varphi_2^*)_*)[\tau_n]$$

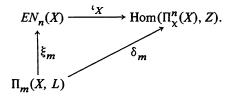$$= \xi_m[\varphi_1] + \xi_m[\varphi_2].$$

The proof of the theorem is complete.

REMARK 4.12. From Corollary 4.7 it follows that if φ is invertible in $C(\partial\Omega) \otimes \mathfrak{M}_n$, then T_{φ^0} is a Fredholm operator in $\mathcal{L}(\mathcal{H})$. In [33] it was shown that if $\Omega = B_m$, and $m \leq n$, then ind $T_{\varphi^0} = (-1/(n-1)!)$ deg $\tilde{\varphi}$, where deg $\tilde{\varphi}$ denotes the degree of $\tilde{\varphi}$ and $\tilde{\varphi}: \partial B_m \to \partial B_m$ is obtained by deforming first the continuous $n \times n$ matrix valued function φ to one of the form $\left(\begin{smallmatrix} \varphi' & 0 \\ 0 & 1_{n-m} \end{smallmatrix}\right)$, where $\varphi' \in C(\partial B_m) \otimes \mathfrak{M}_m$, and then defining $\tilde{\varphi}(z) = \varphi_1'(z)/|\varphi_m'(z)|$, where $\varphi_1'(z)$ is the first column of the matrix $\varphi'(z)$, $z \in \partial B_m$. This deformation can be performed by using the standard argument with the homotopy sequence of the fibration $\mathfrak{U}_k \to \mathfrak{U}_{k+1} \to \partial B_{k+1}$, $k = m, m+1, \ldots$, as mentioned in the proof of Theorem 3.17 (see also [4, p. 239]).

DEFINITION 4.13. For $m \leq n$, $X \in \mathbf{HC}_n$, let $\delta_m: \Pi_m(X, L) \to \mathrm{Hom}(\Pi_X^n(X), Z)$ be the homomorphism given by

$$\delta_m[\varphi]([f]) = (-1/(m-1)!)\deg(\widetilde{f \circ \varphi}).$$

THEOREM 4.14. Let $m \leq n$, $X \in \mathbf{HC}_n$; then the following diagram is commutative:

$$
\begin{array}{ccc}
EN_n(X) & \xrightarrow{\iota_X} & \mathrm{Hom}(\Pi_X^n(X), Z). \\
\uparrow{\scriptstyle \xi_m} & \nearrow{\scriptstyle \delta_m} & \\
\Pi_m(X, L) & &
\end{array}
$$

PROOF. By using the fact that an n-normal operator is quasidiagonal (see for instance [27, Theorem 3.4]), it is easy to see that if S is a Fredholm n-normal operator in $\mathcal{L}(\mathcal{H})$, then its index is zero. Now let $[\varphi] \in \Pi_m(X, L)$ and let $[f] \in \Pi_X^n(X)$. Then, employing [33, Theorem 1.5] we deduce that for

every $[\varphi] \in \Pi_m(X, L)$, $[f] \in \Pi_X^n(X)$, we have

$$((\iota_X \circ \xi_m)[\varphi])[f] = (\iota_X \nu(\varphi^*)_*[\tau_n])[f] = \mathrm{ind}((\pi T_{\varphi^0}, f) \oplus \pi \circ f)$$

$$= \mathrm{ind}(T_{\varphi^0}, f) = (-1/(m-1)!)\deg \widetilde{(\varphi \circ f)} = (\delta_m[\varphi])[f],$$

where $\sigma \in \mathrm{Hom}(C_X^*(X), \mathcal{L}(\mathcal{H}))$ and $[\sigma C_X^*(X)] \cap \mathcal{K}(\mathcal{H}) = \{0\}$. The proof of the theorem is complete.

REMARK 4.15. As pointed out already in Remark 4.9(a), the map ξ_1 is not always injective. On the other hand, from Theorem 4.14 it can be deduced that ξ_1 is not always surjective. Indeed, this is the case when X is the subset of $\mathbf{R}^2$ which is the closure of

$$\{(x, |\sin(\pi/x)|): 0 < x < 1\} \cup \{(x, -\sin \pi x), 0 < x < 1\}.$$

It is reasonable to expect that, for $n > 1$ and some suitable notion for higher order connectivity of a hypoconvex subset X in $\mathfrak{M}_n$, the map ξ_n be an isomorphism. More generally, we believe, that at least in the case in which X is arcwise connected and $EN_n(X) = EN_n^*(X)$, the knowledge of $\Pi_m(X)$, $1 < m < n$, will provide enough information about the hypoconvex set X in $\mathfrak{M}_n$ so that $EN_n(X)$ can be effectively computed for $n > 1$. To this end, the problems to be considered are the following:

(a) Is $\cap_{1 < m < n} \ker \xi_m = \{0\}$?

(b) Given any $[T] \in EN_n(X)$, does there exist a function $\varphi \in C(\partial B_m) \otimes \mathfrak{M}_n$, for $1 < m < n$, and an n-normal operator S in $\mathcal{L}(\mathcal{H})$ with $R_e^n(S) = X$, such that $T \sim T_{\varphi^0} \oplus S$?

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